Title
Asymptotic expansions in time for rotating incompressible viscous fluids

Journal
Annales de l'Institut Henri Poincare (C) Analyse Non Lineaire, 38(1)

ISSN
0294-1449

Authors
Hoang, LT
Titi, ES

Publication Date
2021

DOI
10.1016/j.anihpc.2020.06.005

Peer reviewed

ASYMPTOTIC EXPANSIONS IN TIME FOR ROTATING INCOMPRESSIBLE VISCOUS FLUIDS

LUAN T. HOANG[1] AND EDRISS S. TITI[2]

Abstract. We study the three-dimensional Navier–Stokes equations of rotating incompressible viscous fluids with periodic boundary conditions. The asymptotic expansions, as time goes to infinity, are derived in all Gevrey spaces for any Leray-Hopf weak solutions in terms of oscillating, exponentially decaying functions. The results are established for all non-zero rotation speeds, and for both cases with and without the zero spatial average of the solutions. Our method makes use of the Poincaré waves to rewrite the equations, and then implements the Gevrey norm techniques to deal with the resulting time-dependent bi-linear form. Special solutions are also found which form infinite dimensional invariant linear manifolds.

Contents

1. Introduction

We study the long-time behavior of the three-dimensional incompressible viscous fluids rotated about the vertical axis with a constant angular speed. The Navier–Stokes equations (NSE) written in the rotating frame are used to describe the fluid dynamics in this case, see, e.g., [28]. We denote by $\mathbf{x} \in \mathbb{R}^3$ the spatial variables, $t \in \mathbb{R}_+ = [0,\infty)$ the time variable, and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard canonical basis of $\mathbb{R}^3$. The NSE for the rotating fluids are

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \Omega\mathbf{e}_3 \times \mathbf{u} = 0, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.2}$$

Date: October 21, 2019.
2010 *Mathematics Subject Classification.* 35Q30, 76D05, 35C20, 76E07.
Key words and phrases. Navier-Stokes equations, rotating fluids, asymptotic expansions, long-time dynamics.

where $\mathbf{u}(\mathbf{x},t)$ is the velocity field, p is the pressure adjusted by the fluid's constant density, gravity and centrifugal force, $\nu > 0$ is the kinematic viscosity, and $\frac{1}{2}\Omega\mathbf{e}_3$ is the angular velocity of the rotation.

Above, $\Omega\mathbf{e}_3 \times \mathbf{u}$ represents the Coriolis force exerted on the fluid. We will write

$$\mathbf{e}_3 \times \mathbf{u} = J\mathbf{u}, \text{ where } J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equations (1.1) and (1.2) comprise a system of nonlinear partial differential equations with the unknowns $\mathbf{u}$ and p, while the constants $\nu > 0$ and Ω are given. We will study this system within the context of spatially periodic functions.

Let $L_1, L_2, L_3 > 0$ be the spatial periods and denote $\mathbf{L} = (L_1, L_2, L_3)$. A function $f : \mathbb{R}^3 \to \mathbb{R}^m$, for some $m \in \mathbb{N}$, is $\mathbf{L}$-periodic if

$$f(\mathbf{x} + L_j\mathbf{e}_j) = f(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^3,\ j = 1,2,3.$$

Consider the $\mathbf{L}$-periodic solutions $(\mathbf{u}, p)$, that is, $\mathbf{u}(\cdot,t)$ and $p(\cdot,t)$ are $\mathbf{L}$-periodic for all $t > 0$.

Let $L_* = \max\{L_1, L_2, L_3\}$ and $\lambda_1 = (2\pi/L_*)^2$. Under the transformation

$$\mathbf{u}(\mathbf{x},t) = \lambda_1^{1/2}\nu\,\mathbf{v}(\lambda_1^{1/2}\mathbf{x}, \lambda_1\nu t), \quad p(\mathbf{x},t) = \lambda_1\nu^2 q(\lambda_1^{1/2}\mathbf{x}, \lambda_1\nu t), \quad \Omega = \lambda_1\nu\omega,$$

where $\mathbf{v}(\mathbf{y},\tau)$ and $q(\mathbf{y},\tau)$ are $\lambda_1^{1/2}\mathbf{L}$-periodic adimensional functions, system (1.1) and (1.2) becomes

$$\lambda_1^{3/2}\nu^2\Big(\frac{\partial \mathbf{v}}{\partial \tau} - \Delta_{\mathbf{y}}\mathbf{v} + (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{v} + \nabla_{\mathbf{y}} q + \omega J\mathbf{v}\Big) = 0 \text{ and } \lambda_1\nu \operatorname{div}_{\mathbf{y}}\mathbf{v} = 0,$$

thus,

$$\frac{\partial \mathbf{v}}{\partial \tau} - \Delta_{\mathbf{y}}\mathbf{v} + (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{v} + \nabla_{\mathbf{y}} q + \omega J\mathbf{v} = 0 \text{ and } \operatorname{div}_{\mathbf{y}}\mathbf{v} = 0. \tag{1.3}$$

Thanks to (1.3), we can assume hereafter, without loss of generality, that the Navier–Stokes system (1.1) and (1.2) has

$$\nu = 1, \quad L_* = 2\pi, \quad \lambda_1 = 1. \tag{1.4}$$

In dealing with $\mathbf{L}$-periodic functions, it is convenient to formulate the equations and functional spaces using the domain (the three-dimensional flat torus)

$$\mathbb{T}_{\mathbf{L}} \stackrel{\text{def}}{=} (\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z}) \times (\mathbb{R}/L_3\mathbb{Z}).$$

Regarding the notation in this paper, a vector in $\mathbb{C}^3$ is viewed as a column vector, and we denote the dot product between $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3$ by

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{y}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{y}.$$

Hence, the standard inner product in $\mathbb{C}^3$ is $\mathbf{x}\cdot\bar{\mathbf{y}}$.

For $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$, denote

$$\check{\mathbf{k}} = (\check{k}_1, \check{k}_2, \check{k}_3) \stackrel{\text{def}}{=} 2\pi(k_1/L_1, k_2/L_2, k_3/L_3), \tag{1.5}$$

and, in case $\mathbf{k} \neq \mathbf{0}$,

$$\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, \widetilde{k}_3) \stackrel{\text{def}}{=} \check{\mathbf{k}}/|\check{\mathbf{k}}|, \tag{1.6}$$

$$X_{\mathbf{k}} \stackrel{\text{def}}{=} \{\mathbf{z} \in \mathbb{C}^3 : \mathbf{z}\cdot\check{\mathbf{k}} = 0\} = \{\mathbf{z} \in \mathbb{C}^3 : \mathbf{z}\cdot\widetilde{\mathbf{k}} = 0\}. \tag{1.7}$$

In the following, we present the functional setting and functional formulations for the NSE. We refer the reader to the books [5, 13, 22, 26, 27] for more details.

Denote the inner product and norm in $L^2(\mathbb{T}_{\mathbf{L}})^3$ by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. The latter notation is also used for the modulus of a complex number and the length of a vector in $\mathbb{C}^n$, but its meaning will be clear from the context.

Each $\mathbf{u}\in L^2(\mathbb{T}_{\mathbf{L}})^3$ has the Fourier series

$$\mathbf{u}(\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{Z}^3}\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}},\tag{1.8}$$

where $i=\sqrt{-1}$, $\widehat{\mathbf{u}}_{\mathbf{k}}\in\mathbb{C}^3$ are the Fourier coefficients with the reality condition $\widehat{\mathbf{u}}_{-\mathbf{k}}=\overline{\widehat{\mathbf{u}}_{\mathbf{k}}}$. If $\mathbf{u}$ has zero spatial average over $\mathbb{T}_{\mathbf{L}}$ then $\widehat{\mathbf{u}}_{\mathbf{0}}=0$.

We now focus on the case of solutions $\mathbf{u}(\mathbf{x},t)$ with zero spatial average over $\mathbb{T}_{\mathbf{L}}$, for any $t\geq 0$. The general case will be studied in section 4. For brevity, denote

$$\sum{}'=\sum_{\mathbf{k}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}}.$$

Let $\mathcal{V}$ be the space of zero-average, divergence-free $\mathbf{L}$-periodic trigonometric polynomial vector fields, that is, it consists of functions

$$u=\sum{}'\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}$$

where $\widehat{\mathbf{u}}_{\mathbf{k}}\in X_{\mathbf{k}}$, $\widehat{\mathbf{u}}_{-\mathbf{k}}=\overline{\widehat{\mathbf{u}}_{\mathbf{k}}}$ for all $\mathbf{k}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}$, and $\widehat{\mathbf{u}}_{\mathbf{k}}\neq\mathbf{0}$ for only finitely many $\mathbf{k}$'s.

Let H, respectively (resp.) V, be the closure of $\mathcal{V}$ in $L^2(\mathbb{T}_{\mathbf{L}})^3$, resp. $H^1(\mathbb{T}_{\mathbf{L}})^3$.

We use the following embeddings and identification

$$V\subset H=H'\subset V',$$

where each space is dense in the next one, and the embeddings are compact.

Let $\mathcal{P}$ denote the orthogonal (Leray) projection from $L^2(\mathbb{T}_{\mathbf{L}})^3$ onto H. More precisely,

$$\mathcal{P}\Big(\sum_{\mathbf{k}\in\mathbb{Z}^3}\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}\Big)=\sum{}'[\widehat{\mathbf{u}}_{\mathbf{k}}-(\widehat{\mathbf{u}}_{\mathbf{k}}\cdot\widetilde{\mathbf{k}})\widetilde{\mathbf{k}}]e^{i\check{\mathbf{k}}\cdot\mathbf{x}}=\sum_{\mathbf{k}\in\mathbb{Z}^3}\widehat{P}_{\mathbf{k}}\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}=\sum{}'\widehat{P}_{\mathbf{k}}\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}},$$

where $\widehat{P}_{\mathbf{k}}$'s are symmetric 3×3 matrices given by

$$\widehat{P}_{\mathbf{0}}=0,\quad \widehat{P}_{\mathbf{k}}=I_3-\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}}\text{ for }\mathbf{k}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}.\tag{1.9}$$

The Stokes operator A is a bounded linear mapping from V to its dual space V' defined by

$$\langle A\mathbf{u},\mathbf{v}\rangle_{V',V}=\langle\!\langle\mathbf{u},\mathbf{v}\rangle\!\rangle\stackrel{\text{def}}{=}\sum_{j=1}^{3}\langle\frac{\partial\mathbf{u}}{\partial x_j},\frac{\partial\mathbf{v}}{\partial x_j}\rangle,\ \text{for all }\mathbf{u},\mathbf{v}\in V.$$

As an unbounded operator on H, the operator A has the domain $\mathcal{D}(A)=V\cap H^2(\mathbb{T}_{\mathbf{L}})^3$, and, under the current consideration of periodicity conditions,

$$A\mathbf{u}=-\mathcal{P}\Delta\mathbf{u}=-\Delta\mathbf{u}\in H,\ \text{for all }\mathbf{u}\in\mathcal{D}(A).$$

With the Fourier series, this reads as

$$Au=\sum{}'|\check{\mathbf{k}}|^2\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}},\ \text{for }u=\sum{}'\widehat{\mathbf{u}}_{\mathbf{k}}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}\in\mathcal{D}(A).$$

The spectrum of A is known to be

$$\sigma(A)=\big\{|\check{\mathbf{k}}|^2:\ \mathbf{k}\in\mathbb{Z}^3,\mathbf{k}\neq\mathbf{0}\big\},$$

and each $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. We order

$$\sigma(A) = \{\Lambda_n : n \in \mathbb{N}\}, \text{ where the sequence } (\Lambda_n)_{n=1}^\infty \text{ is strictly increasing.}$$

The additive semigroup generated by $\sigma(A)$ is

$$\langle \sigma(A) \rangle \stackrel{\text{def}}{=} \Big\{ \sum_{j=1}^n \alpha_j \,:\, n \in \mathbb{N}, \alpha_j \in \sigma(A) \text{ for } 1 \le j \le n \Big\}. \tag{1.10}$$

The set $\langle \sigma(A) \rangle$ is ordered as a strictly increasing sequence $(\mu_n)_{n=1}^\infty$. Note that $\Lambda_n, \mu_n \to \infty$ as $n \to \infty$, and, by (1.4), $\mu_1 = \Lambda_1 = \lambda_1 = 1$.

For $\Lambda \in \sigma(A)$, we denote by R_Λ the orthogonal projection from H onto the eigenspace of A corresponding to Λ, and set

$$P_\Lambda = \sum_{\lambda \in \sigma(A), \lambda \le \Lambda} R_\lambda.$$

Note that each vector space $P_\Lambda H$ is finite dimensional.

For $\alpha, \sigma \in \mathbb{R}$ and $u = \sum' \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}} \in H$, define

$$A^\alpha u = \sum{}' |\check{\mathbf{k}}|^{2\alpha} \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \quad e^{\sigma A^{1/2}} u = \sum{}' e^{\sigma|\check{\mathbf{k}}|} \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}},$$

and

$$A^\alpha e^{\sigma A^{1/2}} u = \sum{}' |\check{\mathbf{k}}|^{2\alpha} e^{\sigma|\check{\mathbf{k}}|} \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}.$$

For $\alpha, \sigma \ge 0$, the Gevrey spaces are defined by

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) \stackrel{\text{def}}{=} \{u \in H : |u|_{\alpha,\sigma} \stackrel{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\}.$$

In particular, when $\sigma = 0$ the domain of the fractional operator A^α is

$$\mathcal{D}(A^\alpha) = G_{\alpha,0} = \{u \in H : |A^\alpha u| = |u|_{\alpha,0} < \infty\}.$$

Observe that for $\sigma > 0$, $G_{\alpha,\sigma}$ consists of real analytic divergence-free vector fields.

Thanks to the zero-average condition, the norm $|A^{m/2}u|$ is equivalent to $\|u\|_{H^m(\Omega)^3}$ on the space $\mathcal{D}(A^{m/2})$, for $m = 0, 1, 2, \ldots$

Note that $\mathcal{D}(A^0) = H$, $\mathcal{D}(A^{1/2}) = V$, and $\|u\| \stackrel{\text{def}}{=} |\nabla u|$ is equal to $|A^{1/2}u|$, for all $u \in V$. Also, the norms $|\cdot|_{\alpha,\sigma}$ are increasing in α, σ, hence, the spaces $G_{\alpha,\sigma}$ are decreasing in α, σ.

Regarding the nonlinear terms in the NSE, a bounded linear map $B : V \times V \to V'$ is defined by

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{V',V} = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \stackrel{\text{def}}{=} \int_{\mathbb{T}_\mathbf{L}} ((\mathbf{u} \cdot \nabla)\mathbf{v}) \cdot \mathbf{w}\, d\mathbf{x}, \text{ for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

In particular,

$$B(\mathbf{u}, \mathbf{v}) = \mathcal{P}((\mathbf{u} \cdot \nabla)\mathbf{v}), \text{ for all } \mathbf{u}, \mathbf{v} \in \mathcal{D}(A). \tag{1.11}$$

In fact, if $u = \sum' \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}} \in \mathcal{D}(A)$ and $v = \sum' \widehat{\mathbf{v}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}} \in \mathcal{D}(A)$, then $(u \cdot \nabla)v$ has zero spatial average and

$$(u \cdot \nabla)v = \sum{}' \widehat{\mathbf{b}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \text{ where } \widehat{\mathbf{b}}_{\mathbf{k}} = \sum_{\check{\mathbf{m}} + \check{j} = \check{\mathbf{k}}} i(\widehat{\mathbf{u}}_{\mathbf{m}} \cdot \check{\mathbf{k}}) \widehat{\mathbf{v}}_{\mathbf{j}}, \tag{1.12}$$

and, consequently,

$$B(u,v) = \sum{}' \widehat{P}_{\mathbf{k}} \widehat{\mathbf{b}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{1.13}$$

Applying the projection $\mathcal{P}$ of the equation (1.1), we obtain

$$\frac{du}{dt} + Au + B(u,u) + \Omega\mathcal{P}Ju = 0, \tag{1.14}$$

with solution $u \in H$. In the case of non-rotation, i.e. $\Omega = 0$, equation (1.14) is the standard NSE

$$\frac{du}{dt} + Au + B(u,u) = 0. \tag{1.15}$$

Since $\mathcal{P}u = u$, for $u \in H$, equation (1.14) is equivalent to

$$\frac{du}{dt} + Au + B(u,u) + \Omega Su = 0 \tag{1.16}$$

with $u \in H$, where $S = \mathcal{P}J\mathcal{P}$. Equation (1.16) will be the focus of our study.

Note that if u is as in (1.8), then

$$Su = \sum{}' \widehat{P}_{\mathbf{k}} J \widehat{P}_{\mathbf{k}} \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{1.17}$$

We have the following elementary properties:

$$\langle Su, A^\alpha e^{\sigma A^{1/2}} u\rangle = 0, \text{ for all } \alpha,\sigma \ge 0 \text{ and } u \in \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}), \tag{1.18}$$

$$b(u,v,w) = -b(u,w,v) \text{ and } b(u,v,v) = 0, \text{ for all } u,v,w \in V. \tag{1.19}$$

Because of relation (1.18) with $\alpha = \sigma = 0$, the energy balance/inequality for (1.16) is the same as for (1.15). Hence, the following definitions of weak and regular solutions for (1.16) are quite similar to those for (1.15), see, e.g., [5, 13, 27].

Definition 1.1. (a) *A Leray-Hopf weak solution $u(t)$ of* (1.16) *is a mapping from* $[0,\infty)$ *to* H *such that*

$$u \in C([0,\infty), H_{\rm w}) \cap L^2_{\rm loc}([0,\infty), V),\ \ u' \in L^{4/3}_{\rm loc}([0,\infty), V'),$$

and satisfies

$$\frac{d}{dt}\langle u(t), w\rangle + \langle\langle u(t), w\rangle\rangle + b(u(t), u(t), w) + \Omega\langle Su, w\rangle = 0$$

in the distribution sense in $(0,\infty)$ *(in fact in* $L^{4/3}_{\rm loc}([0,\infty))$*), for all* $w \in V$*, and the energy inequality*

$$\frac{1}{2}|u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2 d\tau \le \frac{1}{2}|u(t_0)|^2$$

holds for $t_0 = 0$ *and almost all* $t_0 \in (0,\infty)$*, and for all* $t \ge t_0$*. Here,* $H_{\rm w}$ *denotes the topological vector space* H *with the weak topology.*

(b) *We say a function* $u(t)$ *is a Leray-Hopf weak solution on* $[T,\infty)$ *if* $u(T+\cdot)$ *is a Leray-Hopf weak solution.*

A Leray-Hopf weak solution $u(t)$ *on* $[T_0,\infty)$*, for some* $T_0 \in \mathbb{R}_+$*, is a* regular solution *on* $[T_0, T_0+T)$*, for some* $0 < T \le \infty$*, if*

$$u \in C([T_0,T_0+T), V) \cap L^2_{\rm loc}([T_0,T_0+T), \mathcal{D}(A)),\ \ \text{and } u' \in L^2_{\rm loc}([T_0,T_0+T), H).$$

A global regular solution *is a regular solution on* $[0,\infty)$.

Same as for equation (1.15), the basic questions, for equation (1.16), about the uniqueness of weak solutions, and the global existence of regular solutions are still open, see Theorem 1.3 below. Regarding the second question, it is proved in Babin-Mahalov-Nicolaenko [1] that for any initial data $u_0 \in V$, there is $\Omega_0 > 0$ depending on u_0 such that global regular solutions exist for all $|\Omega| > \Omega_0$. Moreover, it is also showed in [1] that the long-time dynamics of (1.16), with an additional non-potential body force, is close to that of a so-called "$2\frac{1}{2}$-dimensional NSE" when $|\Omega|$ is sufficiently large. However, these basic questions will be bypassed and the mentioned results of [1] will not be needed in this study. It is due to the eventual regularity and decay (to zero) of the solutions, see Proposition 3.4 below. We, instead, will focus on the refined analysis of that decay.

When $\Omega = 0$, the long-time dynamics of equation (1.15), which covers the case of potential forces, is studied in details early in [14–18], and later in [7–11,20]. (The case of non-potential forces is treated in [2, 3, 21]. See also the survey paper [12] for more information.) Briefly speaking, it is proved in [17] that any solution $u(t)$ of (1.15) admits an asymptotic expansion, as $t \to \infty$,

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t}, \tag{1.20}$$

where $q_n(t)$ is an $\mathcal{V}$-valued polynomial in t. See Definition 2.2 below for more information.

When $\Omega \neq 0$, one can initially view equation (1.16) the same as (1.15) with the linear operator $\tilde{A} = A + \Omega S$ replacing A, and follow [17] to obtain the asymptotic expansions. However, the spectrum of $\tilde{A}$, and its eigenspaces will be more complicated. The expansions will be of the form

$$u(t) \sim \sum_{\mu \in \langle \sigma(\tilde{A}) \rangle} q_\mu(t) e^{-\mu t}, \tag{1.21}$$

where $\langle \sigma(\tilde{A}) \rangle$ is defined similarly to (1.10).

One can see that the additive semigroup $\langle \sigma(\tilde{A}) \rangle$ is a set of complex numbers, much more complicated than $\langle \sigma(A) \rangle$, and consequently expansion (1.21) is considerably more complicated than (1.20). The construction of $q_\mu(t)$ must consider various scenarios including resonance and non-resonance cases, which are harder to track for complex values μ's. To avoid all these technicalities, we propose another approach that converts (1.16) to (1.15) with a time-dependent bilinear form. By maintaining the operator A, the expansions will be similar to (1.20), and the proof will be direct and clean in the spirit of [17], taking advantage of recent improvements in [20, 21]. This approach was, in fact, successfully used in [1] in studying the global well-posedness for regular solutions of (1.16).

Rewriting the variational NSE using the Poincaré waves. In dealing with term $\Omega S u$ in (1.16), we will make a change of variables using the exponential operator e^{tS}. Clearly, S is a bounded linear operator on $L^2(\mathbb{T}_{\mathbf{L}})^3$ with norm $\|S\|_{\mathcal{L}(L^2(\mathbb{T}_{\mathbf{L}})^3)} = 1$. For our problem, we restrict e^{tS} to H only, and have the isometry group $e^{tS} : H \to H$, $t \in \mathbb{R}$, is analytic in t.

It is well-known, see e.g. [1, 4], that one has, for $u = \sum{}' \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot \mathbf{x}} \in H$,

$$e^{tS} u = \sum{}' E_{\mathbf{k}}(\widetilde{k}_3 t) \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot \mathbf{x}}, \quad \text{where } E_{\mathbf{k}}(t) = \cos(t) I_3 + \sin(t) J_{\mathbf{k}}, \tag{1.22}$$

with $J_{\mathbf{k}}$ being the 3×3 matrix for which $J_{\mathbf{k}}\mathbf{z}=\widetilde{\mathbf{k}}\times\mathbf{z}$, for all $\mathbf{z}\in\mathbb{C}^3$. Explicitly, the matrix $J_{\mathbf{k}}$, for $\mathbf{k}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}$, is

$$J_{\mathbf{k}}=\begin{pmatrix} 0 & -\widetilde{k}_3 & \widetilde{k}_2 \\ \widetilde{k}_3 & 0 & -\widetilde{k}_1 \\ -\widetilde{k}_2 & \widetilde{k}_1 & 0 \end{pmatrix}. \tag{1.23}$$

For the reader's convenience, we include a simple proof of (1.22) in Appendix A.

One sees that

$$\begin{aligned} &(E_{\mathbf{k}}(t))^*=E_{\mathbf{k}}(-t),\\ &|E_{\mathbf{k}}(t)\mathbf{z}|=|\mathbf{z}|,\ \text{for all } \mathbf{z}\in X_{\mathbf{k}}. \end{aligned}$$

From these properties, we deduce

$$(e^{tS})^*=e^{-tS}\quad (\text{on } H), \tag{1.24}$$

$$|e^{tS}u|_{\alpha,\sigma}=|u|_{\alpha,\sigma},\ \text{for all } \alpha,\sigma\geq 0 \text{ and } u\in\mathcal{D}(A^\alpha e^{\sigma A^{1/2}}). \tag{1.25}$$

Also, some relations between S and A are

$$ASu=SAu,\quad e^{tS}Au=Ae^{tS}u,\ \text{for all } u\in\mathcal{D}(A) \text{ and } t\in\mathbb{R}. \tag{1.26}$$

The bi-linear form in (1.16) will be transformed to a similar, but time-dependent one that we describe below.

Let $t,\Omega\in\mathbb{R}$. Define $b(t,\cdot,\cdot,\cdot):V^3\to\mathbb{R}$ and $b_\Omega(t,\cdot,\cdot,\cdot):V^3\to\mathbb{R}$ by

$$b(t,u,v,w)=b(e^{-tS}u,e^{-tS}v,e^{-tS}w),\quad b_\Omega(t,u,v,w)=b(\Omega t,u,v,w),$$

for all $u,v,w\in V$. We then define $B(t,\cdot,\cdot):V\times V\to V'$ by

$$\langle B(t,u,v),w\rangle_{V',V}=b(t,u,v,w),\ \text{for all } u,v,w\in V. \tag{1.27}$$

In particular, thanks to (1.11) and (1.24),

$$B(t,u,v)=e^{tS}B(e^{-tS}u,e^{-tS}v),\ \text{for all } u,v\in\mathcal{D}(A). \tag{1.28}$$

Define $B_\Omega(t,u,v)=B(\Omega t,u,v)$.

We now rewrite equation (1.16) using the Poincaré waves $e^{-\Omega tS}w$, for $w\in H$ and $t\in\mathbb{R}$. Let $u(t)$ be a solution of (1.16). Set $v(t)=e^{\Omega tS}u(t)$, or equivalently, $u(t)=e^{-\Omega tS}v(t)$. If $u\in C^1((0,\infty),H)\cap C((0,\infty),\mathcal{D}(A))$ then, with (1.26) taken into account, v solves

$$\frac{dv}{dt}+Av+B_\Omega(t,v,v)=0,\quad t>0. \tag{1.29}$$

With this formulation, equation (1.29) resembles more with (1.15) than (1.16). The difference between (1.29) and (1.15) is the time-dependent bi-linear form $B_\Omega(t,\cdot,\cdot)$. However, this bi-linear form turns out to possess many features similar to $B(\cdot,\cdot)$ itself.

For example, from (1.27) and (1.19), we have, for all $\Omega,t\in\mathbb{R}$ and $u,v,w\in V$, that

$$\langle B_\Omega(t,u,v),w\rangle_{V',V}=-\langle B_\Omega(t,u,w),v\rangle_{V',V}, \tag{1.30}$$

and, consequently,

$$\langle B_\Omega(t,u,v),v\rangle_{V',V}=0. \tag{1.31}$$

This prompts the following definition of weak solutions of (1.29).

Definition 1.2. *A Leray-Hopf weak solution $v(t)$ of* (1.29) *is a mapping from $[0,\infty)$ to H such that*

$$v \in C([0,\infty), H_{\rm w}) \cap L^2_{\rm loc}([0,\infty), V), \quad v' \in L^{4/3}_{\rm loc}([0,\infty), V'),$$

and satisfies

$$\frac{d}{dt}\langle v(t), w\rangle + \langle\langle v(t), w\rangle\rangle + b_\Omega(t, v(t), v(t), w) = 0$$

in the distribution sense in $(0,\infty)$ (in fact in $L^{4/3}_{\rm loc}([0,\infty))$), for all $w \in V$, and the energy inequality

$$\frac{1}{2}|v(t)|^2 + \int_{t_0}^t \|v(\tau)\|^2 d\tau \le \frac{1}{2}|v(t_0)|^2$$

holds for $t_0 = 0$ and almost all $t_0 \in (0,\infty)$, and all $t \ge t_0$.

Other definitions in (b) *of Definition 1.1 are extended to the solution $v(t)$.*

Similar to the case $\Omega = 0$, we have the following existence, uniqueness and regularity results for equations (1.16) and (1.29).

Theorem 1.3. *Let $\Omega \neq 0$ be a given number.*

(i) *For any $u_0 \in H$, there exists a Leray-Hopf weak solution $u(t)$ of* (1.16)*, resp. $v(t)$ of* (1.29)*, with initial data u_0. Moreover, there is $T_0 = T_0(u_0) \ge 0$ such that u, resp. v, is a regular solution on $[T_0,\infty)$.*

(ii) *For any $u_0 \in V$, there exists a unique regular solution $u(t)$ of* (1.16)*, resp. $v(t)$ of* (1.29)*, with initial data u_0, on an interval $[0,T)$ for some $T > 0$. If $\|u_0\|$ is sufficiently small, then $T = \infty$.*

(iii) *For any Leray-Hopf weak solution $u(t)$ of* (1.16)*, resp. $v(t)$ of* (1.29)*, and any number $\sigma > 0$, there exists $T_* > 0$ such that $u(t)$, resp. $v(t)$, belongs to $G_{1/2,\sigma}$, for all $t \ge T_*$, and the equation* (1.16)*, resp.* (1.29)*, holds in $\mathcal{D}(A)$ on (T_*,∞) with classical time derivative.*

Parts (i) and (ii) of Theorem 1.3 are standard. Part (iii) can be proved by using the same technique of Foias-Temam [19], noticing that we have the orthogonality (1.18). See details of similar calculations in [20], and also more specific statements in Proposition 3.4 below.

The current paper is focused on a different question, namely, the precise long-time dynamics of the solutions of (1.16) for all $\Omega \neq 0$. Even though they eventually, as $t \to \infty$, go to zero, our goal is to provide a detailed description of such a decay. In the case with the zero spatial average of the solutions, we will show that each solution possesses an asymptotic expansion which is similar to (1.20), but contains some oscillating terms that are from the rotation. The oscillating parts, in this case, are written in terms of sinusoidal functions of time. (See a similar result by Shi [25] for dissipative wave equations.) In the general case of non-zero spatial average, similar asymptotic expansions are obtained with the oscillating terms being expressed by the "double sinusoidal" functions.

This paper is organized as follows. In section 2, we introduce our abstract asymptotic expansions, for large time, in terms of oscillating-decaying functions. Our basic classes of functions are the S-polynomials and SS-polynomials, see Definition 2.1, below. Fundamental properties of these two classes are studied. In particular, when the forcing term of a linear ordinary differential equation (ODE) is a perturbation of an S-polynomial, then the ODE's solution can be approximated by S-polynomials, see Lemma 2.9, below. This lemma turns out to be a building block in our constructions of polynomials in the asymptotic expansions.

In section 3, we establish the asymptotic expansions for solutions of (1.29) and (1.16), below, in Theorems 3.1 and 3.2, respectively. The expansions are in terms of S-polynomials and exponential functions. It is worth mentioning that these results are established for *all* $\Omega \neq 0$ and for all Leray-Hopf weak solutions. In particular, it does not rely on Babin-Mahalov-Nicolaenko's global well-posedness result [1], which, as mentioned after Definition 1.1, requires Ω to be *sufficiently large* depending on the initial data. In section 4, we derive the asymptotic expansions for solutions without the zero spatial averages. This is done by using the specific Galilean-type transformation (4.5). Unlike the previous section, this yields the expansions in terms of SS-polynomials. In section 5, we present some special solutions that form infinite dimensional linear manifolds that are invariant under the flows generated by the solutions of (1.16). Especially, Remark 5.2 contains a subclass of solutions for which the helicity, a meaningful physical quantity in fluid dynamics [23, 24], vanishes. The case of non-zero spatial average of the solutions is treated in Theorem 5.4, below.

2. Abstract asymptotic expansions and their properties

We introduce here the classes of functions which will appear in our asymptotic expansions.

Definition 2.1. *Let X be a vector space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

(a) *A function $g : \mathbb{R} \to X$ is an X-valued S-polynomial if it is a finite sum of the functions of the set*

$$\Big\{t^m \cos(\omega t)Z,\ t^m \sin(\omega t)Z : m \in \mathbb{N} \cup \{0\},\ \omega \in \mathbb{R},\ Z \in X\Big\}. \tag{2.1}$$

(b) *A function $g : \mathbb{R} \to X$ is an X-valued SS-polynomial if it is a finite sum of the functions of the set*

$$\begin{aligned} \Big\{ & t^m \cos\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z, \\ & t^m \sin\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z : \\ & m \in \mathbb{N} \cup \{0\},\ a, b, c, d, \omega \in \mathbb{R},\ Z \in X\Big\}. \end{aligned} \tag{2.2}$$

(c) *Denote by $\mathcal{F}_0(X)$, resp., $\mathcal{F}_1(X)$ and $\mathcal{F}_2(X)$ the set of all X-valued polynomials, resp., S-polynomials and SS-polynomials.*

Clearly, $\mathcal{F}_0(X)$, $\mathcal{F}_1(X)$ and $\mathcal{F}_2(X)$ are vector spaces over $\mathbb{K}$, and $\mathcal{F}_0(X) \subset \mathcal{F}_1(X) \subset \mathcal{F}_2(X)$. If $f \in \mathcal{F}_1(X)$, then we can write f as

$$f(t) = \sum_{n=0}^{N} t^n f_n(t), \tag{2.3}$$

where

$$f_n(t) = \sum_{j=1}^{N_n} [a_{n,j} \cos(\omega_{n,j} t) + b_{n,j} \sin(\omega_{n,j} t)], \tag{2.4}$$

with $N_n \in \mathbb{N}$, $a_{n,j}, b_{n,j} \in X$, and $\omega_{n,j} \geq 0$ with the mapping $j \mapsto \omega_{n,j}$ being strictly increasing for each fixed n.

Definition 2.2. *Let $(X, \|\cdot\|_X)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative real numbers. Let $\mathcal{F} = \mathcal{F}_0$, $\mathcal{F}_1$, or $\mathcal{F}_2$. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}_+$, is said to have an asymptotic expansion*

$$f(t) \sim \sum_{n=1}^\infty f_n(t) e^{-\alpha_n t} \quad \text{in } X, \tag{2.5}$$

where each f_n belongs to $\mathcal{F}(X)$, if one has, for any $N \geq 1$, that

$$\Big\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \Big\|_X = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}), \text{ as } t \to \infty, \tag{2.6}$$

for some $\varepsilon_N > 0$.

With this definition, the precise statement of (1.20) is that it holds, with $\mathcal{F}(X) = \mathcal{F}_0(X)$, in the space $X = G_{\alpha,\sigma}$, for all $\alpha, \sigma \geq 0$.

The expansion (2.5) with $\mathcal{F} = \mathcal{F}_1$ is equivalent to the one used in [25], for the dissipative wave equations, while with the largest class $\mathcal{F} = \mathcal{F}_2$ is new. For other related asymptotic expansions for solutions of NSE, see [2, 3].

One can observe, same as in [21, Remark 2.3], that if (2.6) holds for all N then

$$\Big\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \Big\|_X = \mathcal{O}(e^{-\alpha t}), \text{ as } t \to \infty, \tag{2.7}$$

for all N and all $\alpha \in (\alpha_N, \alpha_{N+1})$.

Lemma 2.3. *Given $N \in \mathbb{N}$, if $f_1, f_2, \ldots, f_N \in \mathcal{F}_1(X)$ satisfy* (2.6)*, then such S-polynomials f_n's are unique.*

Proof. Let $g_1, g_2, \ldots, g_N$ be functions in $\mathcal{F}_1(X)$ that satisfy

$$\Big\| f(t) - \sum_{n=1}^N g_n(t) e^{-\alpha_n t} \Big\|_X = \mathcal{O}(e^{-(\alpha_N + \varepsilon'_N)t}), \text{ as } t \to \infty \text{ for some } \varepsilon'_N > 0.$$

Let $h_n = f_n - g_n$. By the triangle inequality, we have

$$\Big\| \sum_{n=1}^N h_n(t) e^{-\alpha_n t} \Big\|_X \leq \Big\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \Big\|_X + \Big\| f(t) - \sum_{n=1}^N g_n(t) e^{-\alpha_n t} \Big\|_X,$$

hence

$$\Big\| \sum_{n=1}^N h_n(t) e^{-\alpha_n t} \Big\|_X = \mathcal{O}(e^{-(\alpha_N + \varepsilon)t}), \text{ as } t \to \infty, \text{ for } \varepsilon = \min\{\varepsilon_N, \varepsilon'_N\} > 0. \tag{2.8}$$

Suppose that not all h_n's are zero functions. Let n_0 be the smallest number such that $h_{n_0} \neq 0$. By multiplying (2.8) with $e^{\alpha_{n_0} t}$, we deduce

$$\|h_{n_0}(t)\|_X = \mathcal{O}(e^{-\varepsilon' t}), \text{ for some } \varepsilon' > 0. \tag{2.9}$$

We write $h_{n_0}(t)$ in the form of (2.3). Suppose the highest order term (with respect to the power of t) of $h_{n_0}(t)$ is $t^d z(t)$, where d is a non-negative integer, and z is a non-zero function of the form as in the RHS of (2.4). Then dividing (2.9) by t^d yields

$$\lim_{t\to\infty} \|z(t)\|_X = 0. \tag{2.10}$$

Suppose

$$z(t) = \sum_{n=1}^{N_d} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)], \tag{2.11}$$

where $N_d \in \mathbb{N}$, $a_n, b_n \in X$ and $\omega_n \in \mathbb{R}$, for $1 \le n \le N_d$.

Let Y be the vector space spanned by $\{a_n, b_n \in X : 1 \le n \le N_d\}$. Let $\mathcal{Y} = \{Y_j : 1 \le j \le m\}$ be a basis of Y. By representing the vectors a_n's and b_n's in basis $\mathcal{Y}$, we can rewrite $z(t)$ as

$$z(t) = \sum_{j=1}^{m} z_j(t) Y_j,$$

where $m \in \mathbb{N}$, each $z_j : \mathbb{R} \to \mathbb{K}$, for $1 \le j \le m$, is a linear combination of the functions $\cos(\omega_n t)$ and $\sin(\omega_n t)$, for $1 \le n \le N_d$, in (2.11).

For $y = \sum_{j=1}^m y_j Y_j \in Y$, with $y_j \in \mathbb{K}$ for $1 \le j \le m$, define the norm

$$\|y\|_Y = \Big(\sum_{j=1}^{m} |y_j|^2 \Big)^{1/2}.$$

On the finite dimensional space Y, the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent. Therefore, (2.10) gives

$$\lim_{t\to\infty} \|z(t)\|_Y = 0, \text{ which implies } \lim_{t\to\infty} z_j(t) = 0, \text{ for } 1 \le j \le m.$$

By Lemma A.1, we obtain $z_j = 0$ for all j. Hence $z = 0$, which leads to a contradiction. Thus, $h_n = 0$, for all $n = 1, 2, \ldots, N$. We conclude $f_n = g_n$, for $1 \le n \le N$. The proof is complete. □

Remark 2.4. We discuss a consequence of Lemma 2.3. Suppose (2.6) is satisfied with $X = G_{\beta_i,\sigma_i}$ and $f_n = g_n^{(i)} \in \mathcal{F}_1(X)$, for $i = 1, 2$ and $1 \le n \le N$. Let $\bar\beta = \min\{\beta_1, \beta_2\}$ and $\bar\sigma = \min\{\sigma_1, \sigma_2\}$. Then (2.6) is satisfied with both $f_n = g_n^{(1)}$ and $f_n = g_n^{(2)}$ on the same space $X = G_{\bar\beta,\bar\sigma}$. By the virtue of Lemma 2.3, we have $g_n^{(1)} = g_n^{(2)}$, for $1 \le n \le N$.

2.1. **Properties of S-polynomials and SS-polynomials.** We start this subsection with some elementary properties of the functions introduced in Definition 2.1 above.

Lemma 2.5. *Let X and $\mathbb{K}$ be as in Definition 2.1, and $\mathcal{F}(X) = \mathcal{F}_1(X)$ or $\mathcal{F}(X) = \mathcal{F}_2(X)$. Let f be any function in $\mathcal{F}(X)$.*

(i) *The functions $t \mapsto f(T+t)$ and $t \mapsto f(kt)$ belong to $\mathcal{F}(X)$ for any numbers $T, k \in \mathbb{R}$.*
(ii) *If $g \in \mathcal{F}_1(\mathbb{K})$, then $gf \in \mathcal{F}(X)$.*
(iii) *In case X is a subspace of $\mathbb{C}^n$, with $\mathbb{K} = \mathbb{C}$, then $e^{i\omega t} f \in \mathcal{F}(X)$.*
(iv) *Suppose Y is another vector space over $\mathbb{K}$ and L is a linear mapping from X to Y. Then $Lf \in \mathcal{F}(Y)$.*

Proof. Parts (i) and (ii) can be easily verified by using elementary trigonometric identities such as the sine and cosine of a sum, and the product to sum formulas. Part (iii) is obtained by applying part (ii) to the function $g(t) = e^{i\omega t}$ which, in fact, belongs to $\mathcal{F}_1(\mathbb{C})$. Part (iv) is obvious. □

Lemma 2.6. *Let $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$. Then, a function f belongs to $\mathcal{F}(\mathcal{V})$ if and only if*

$$f(t) = \sideset{}{'}\sum_{\text{finitely many } \mathbf{k}} \mathbf{f_k}(t) e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \ \textit{with each} \quad \mathbf{f_k} \in \mathcal{F}(X_\mathbf{k}), \ \textit{and } \mathbf{f_{-k}} = \overline{\mathbf{f_k}}, \tag{2.12}$$

where $X_{\mathbf{k}}$ *is as in* (1.7).

Proof. We prove for the case $\mathcal{F}=\mathcal{F}_1$. The arguments for the other case $\mathcal{F}=\mathcal{F}_2$ are similar and omitted.

Suppose $f\in\mathcal{F}_1(\mathcal{V})$. We write

$$f(t)=\sum_{j=1}^{N} t^{m_j}[a_j\cos(\omega_j t)+b_j\sin(\omega_j t)]u_j,\ \text{each } a_j,b_j\in\mathbb{R},\ u_j\in\mathcal{V}.$$

By writing the finite Fourier series of each u_j and combining the coefficients for $e^{i\check{\mathbf{k}}\cdot\mathbf{x}}$, we find that the Fourier series of f is of the form as in (2.12) with each $\mathbf{f}_{\mathbf{k}}(t)$ being a finite sum of $t^{m_j}[a_j\cos(\omega_j t)+b_j\sin(\omega_j t)]\mathbf{z}$ for some $\mathbf{z}\in X_{\mathbf{k}}$. Thus, $\mathbf{f}_{\mathbf{k}}\in\mathcal{F}_1(X_{\mathbf{k}})$. The last relation in (2.12) is the standard condition for f to be real-valued.

Now, suppose f is as in (2.12). For each $\mathbf{k}$, consider the function

$$F_{\mathbf{k}}(t)=\mathbf{f}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}}+\mathbf{f}_{-\mathbf{k}}(t)e^{-i\check{\mathbf{k}}\cdot\mathbf{x}}=\mathbf{f}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}}+\overline{\mathbf{f}_{\mathbf{k}}(t)}e^{-i\check{\mathbf{k}}\cdot\mathbf{x}}.$$

If $\mathbf{f}_{\mathbf{k}}(t)$ contains $t^m\cos(\omega t)\mathbf{z}$ for some $\mathbf{z}\in X_{\mathbf{k}}$, then $F_{\mathbf{k}}(t)$ contains

$$t^m\cos(\omega t)(\mathbf{z}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}+\bar{\mathbf{z}}e^{-i\check{\mathbf{k}}\cdot\mathbf{x}}).\tag{2.13}$$

Since $\mathbf{z}e^{i\check{\mathbf{k}}\cdot\mathbf{x}}+\bar{\mathbf{z}}e^{-i\check{\mathbf{k}}\cdot\mathbf{x}}\in\mathcal{V}$, the function in (2.13) belongs to $\mathcal{F}_1(\mathcal{V})$. Similar property holds for $\sin(\omega t)$ replacing $\cos(\omega t)$, and we obtain $F_{\mathbf{k}}\in\mathcal{F}_1(\mathcal{V})$. Then f being a finite sum of such $F_{\mathbf{k}}$'s yields $f\in\mathcal{F}_1(\mathcal{V})$. ☐

The following are important properties relating the S- and SS- polynomials with the rotation and nonlinear terms in the rotational NSE.

Lemma 2.7. *Let* $\Lambda\in\sigma(A)$*, two functions* $f,g\in\mathcal{F}_1(P_\Lambda H)$*, and* $\Omega\in\mathbb{R}$*. Then*

$$e^{\Omega tS}f(t)\in\mathcal{F}_1(P_\Lambda H),\tag{2.14}$$

$$B(f(t),g(t))\in\mathcal{F}_1(P_{4\Lambda}H),\tag{2.15}$$

$$B_\Omega(t,f(t),g(t))\in\mathcal{F}_1(P_{4\Lambda}H).\tag{2.16}$$

Proof. (a) By Lemma 2.6, we can write $f(t)$ as

$$f(t)=\sum_{|\check{\mathbf{k}}|^2\le\Lambda}{}' \mathbf{f}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}},\ \text{with each } \mathbf{f}_{\mathbf{k}}\in\mathcal{F}_1(X_{\mathbf{k}}),\ \text{and } \mathbf{f}_{-\mathbf{k}}=\overline{\mathbf{f}_{\mathbf{k}}}.\tag{2.17}$$

Applying (1.22) yields

$$e^{\Omega tS}f(t)=\sum_{|\check{k}|^2\le\Lambda}{}' E_{\mathbf{k}}(\Omega\widetilde{k}_3 t)\mathbf{f}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}}.\tag{2.18}$$

Applying Lemma 2.5 to $\mathcal{F}=\mathcal{F}_1$, $g(t):=\cos(\Omega\widetilde{k}_3 t)$ and then $g(t):=\sin(\Omega\widetilde{k}_3 t)$, one has each $E_{\mathbf{k}}(\Omega\widetilde{k}_3 t)\mathbf{f}_{\mathbf{k}}(t)$ belongs to $\mathcal{F}_1(X_{\mathbf{k}})$. Then by the virtue of the sufficient condition in Lemma 2.6, we have $e^{\Omega tS}f(t)\in\mathcal{F}_1(\mathcal{V})$. This and the restriction $|\check{\mathbf{k}}|^2\le\Lambda$ in (2.18) give (2.14).

(b) We prove (2.15). By Lemma 2.6 again, we can assume, in addition to (2.17), that

$$g(t)=\sum_{|\check{\mathbf{k}}|^2\le\Lambda}{}' \mathbf{g}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}},\ \text{with each } \mathbf{g}_{\mathbf{k}}\in\mathcal{F}_1(X_{\mathbf{k}}),\ \text{and } \mathbf{g}_{-\mathbf{k}}=\overline{\mathbf{g}_{\mathbf{k}}}.$$

By (1.13), we have

$$B(f(t),g(t)) = \sum{}' \mathbf{B}_{\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \tag{2.19}$$

where

$$\mathbf{B}_{\mathbf{k}}(t) = \sum_{\substack{0<|\check{\mathbf{m}}|^2,|\check{\mathbf{j}}|^2\le\Lambda,\\ \check{\mathbf{m}}+\check{\mathbf{j}}=\check{\mathbf{k}}}} (\mathbf{f}_{\mathbf{m}}(t)\cdot i\,\check{\mathbf{k}})\widehat{P}_{\mathbf{k}}\mathbf{g}_{\mathbf{j}}(t). \tag{2.20}$$

Thanks to formula (2.20) for $\mathbf{B}_{\mathbf{k}}$, we only need to sum over $\mathbf{k}$ in (2.19) with

$$|\check{\mathbf{k}}|^2 = |\check{\mathbf{m}}+\check{\mathbf{j}}|^2 \le 2(|\check{\mathbf{m}}|^2+|\check{\mathbf{j}}|^2) \le 4\Lambda.$$

Thus, $B(f(t),g(t)) \in P_{4\Lambda}H$ for all $t\in\mathbb{R}$.

Note that $\widehat{P}_{\mathbf{k}} = \widehat{P}_{-\mathbf{k}}$. In the sum (2.20), we will pair $\check{\mathbf{m}}+\check{\mathbf{j}} = \check{\mathbf{k}}$ for $\mathbf{B}_{\mathbf{k}}(t)$ with $(-\check{\mathbf{m}})+(-\check{\mathbf{j}}) = (-\check{\mathbf{k}})$ for $\mathbf{B}_{-\mathbf{k}}(t)$. Because $\big(\mathbf{f}_{\mathbf{m}}\cdot i\,\check{\mathbf{k}}\big)\widehat{P}_{\mathbf{k}}\mathbf{g}_{\mathbf{j}}$, for $\mathbf{B}_{\mathbf{k}}(t)$, and $\big(\mathbf{f}_{-\mathbf{m}}\cdot(-i\,\check{\mathbf{k}})\big)\widehat{P}_{-\mathbf{k}}\mathbf{g}_{-\mathbf{j}}$, for $\mathbf{B}_{-\mathbf{k}}(t)$, are conjugates of each other, so are $\mathbf{B}_{\mathbf{k}}(t)$ and $\mathbf{B}_{-\mathbf{k}}(t)$.

Using Lemma 2.5 (ii) and (iv), one can verify that $\big(\mathbf{f}_{\mathbf{m}}\cdot i\,\check{\mathbf{k}}\big)\widehat{P}_{\mathbf{k}}\mathbf{g}_{\mathbf{j}}$ belongs to $\mathcal{F}_1(X_{\mathbf{k}})$, hence $\mathbf{B}_{\mathbf{k}}(t)\in\mathcal{F}_1(X_{\mathbf{k}})$.

By combining the above facts with Lemma 2.6, we conclude (2.15).

(c) Next, we prove (2.16). Because $f(t),g(t)\in\mathcal{F}_1(P_\Lambda H)$, we apply (2.14) to have $e^{-\Omega tS}f(t)$ and $e^{-\Omega tS}g(t)$ belong to $\mathcal{F}_1(P_\Lambda H)$, which, by (2.15), imply

$$B(e^{-\Omega tS}f(t),e^{-\Omega tS}g(t))\in\mathcal{F}_1(P_{4\Lambda}H),$$

which, in turn, thanks to (2.14) again, further implies

$$e^{\Omega tS}B(e^{-\Omega tS}f(t),e^{-\Omega tS}g(t))\in\mathcal{F}_1(P_{4\Lambda}H).$$

Therefore, thanks also to (1.28), we conclude (2.16). □

2.2. **Approximating solutions of certain linear ODEs with S-polynomials.** In our proofs, we often need the following integrals.

Lemma 2.8. *Let $\alpha,\omega\in\mathbb{R}$ with $\alpha^2+\omega^2>0$, and m be a non-negative integer. Then each integral*

$$\int t^m e^{\alpha t}\cos(\omega t)dt,\quad \int t^m e^{\alpha t}\sin(\omega t)dt \tag{2.21}$$

is of the form

$$p(t)e^{\alpha t}\cos(\omega t)+q(t)e^{\alpha t}\sin(\omega t)+const.,$$

where $p(t)$ and $q(t)$ are polynomials of degrees at most m.

Although this lemma is elementary, we give a proof in Appendix A that yields simple and explicit formulas for the integrals in (2.21), see (A.4) below.

The next lemma essentially originates from Foias-Saut [17], but is stated and proved in the same convenient form as [21, Lemma 4.2].

Lemma 2.9. *Let $(X,\|\cdot\|_X)$ be a Banach space. Suppose y is a function in $C([0,\infty),X)$, with distribution derivative $y'\in L^1_{\rm loc}([0,\infty),X)$, that solves the following ODE*

$$y'(t)+\beta y(t) = p(t)+g(t)$$

in the X-valued distribution sense on $(0,\infty)$, where $\beta\in\mathbb{R}$ is a constant, $p(t)$ is an X-valued S-polynomial, and $g\in L^1([0,\infty),X)$ satisfies

$$\|g(t)\|_X \le Me^{-\delta t},\ \text{for all } t\ge 0,\ \text{and some } M,\delta>0. \tag{2.22}$$

Define $q(t)$, *for* $t\in\mathbb{R}$, *by*

$$q(t)=\begin{cases}e^{-\beta t}\int_{-\infty}^t e^{\beta\tau}p(\tau)d\tau & \text{if } \beta>0,\\ y(0)+\int_0^\infty g(\tau)d\tau+\int_0^t p(\tau)d\tau & \text{if } \beta=0,\\ -e^{-\beta t}\int_t^\infty e^{\beta\tau}p(\tau)d\tau & \text{if } \beta<0.\end{cases} \tag{2.23}$$

Then $q(t)$ *is an* X*-valued S-polynomial that satisfies*

$$q'(t)+\beta q(t)=p(t),\ \text{for all } t\in\mathbb{R}, \tag{2.24}$$

and the following estimates hold:

(i) *If* $\beta>0$ *then*

$$\|y(t)-q(t)\|_X^2\le 2e^{-2\beta t}\|y(0)-q(0)\|_X^2+2t\int_0^t e^{-2\beta(t-\tau)}\|g(\tau)\|_X^2d\tau,\ \text{for all } t\ge 0. \tag{2.25}$$

(ii) *If either*

(a) $\beta=0$, *or*

(b) $\beta<0$ *and*

$$\lim_{t\to\infty}(e^{\beta t}\|y(t)\|_X)=0, \tag{2.26}$$

then

$$\|y(t)-q(t)\|_X^2\le\Big(\frac{M}{\delta-\beta}\Big)^2e^{-2\delta t},\qquad \text{for all } t\ge 0. \tag{2.27}$$

Proof. Thanks to Lemma 2.8, $q(t)$ is an X-valued S-polynomial. The rest of this lemma is the same as [21, Lemma 4.2], except for the relaxed estimate (2.25) which we verify now. Consider $\beta>0$. Let $z(t)=y(t)-q(t)$. Recall the inequality after (4.13) in [21, Lemma 4.2], for all $t\ge 0$,

$$\|z(t)\|_X\le e^{-\beta t}\|z(0)\|+\int_{t_0}^t e^{-\beta(t-\tau)}\|g(\tau)\|_Xd\tau. \tag{2.28}$$

Using Cauchy-Schwarz's and Hölder's inequalities, we estimate

$$\begin{aligned}\|z(t)\|_X^2&\le 2e^{-2\beta t}\|z(0)\|_X^2+2\Big(\int_0^t e^{-\beta(t-\tau)}\|g(\tau)\|_Xd\tau\Big)^2\\ &\le 2e^{-2\beta t}\|z(0)\|_X^2+2t\int_0^t e^{-2\beta(t-\tau)}\|g(\tau)\|_X^2d\tau.\end{aligned}$$

Therefore, we obtain (2.25). ∎

3. The case of zero spatial average solutions

We obtain two main asymptotic expansion results, one for equation (1.29), and the other for equation (1.16).

Theorem 3.1. *For any Leray-Hopf weak solution* $v(t)$ *of* (1.29), *there exist unique* $\mathcal{V}$*-valued S-polynomials* q_n*'s, for all* $n\in\mathbb{N}$, *such that it holds, for any* $\alpha,\sigma>0$ *and* $N\ge 1$, *that*

$$\Big|v(t)-\sum_{n=1}^N q_n(t)e^{-\mu_n t}\Big|_{\alpha,\sigma}=\mathcal{O}\big(e^{-\mu t}\big),\ \text{as } t\to\infty,\ \text{for all } \mu\in(\mu_N,\mu_{N+1}). \tag{3.1}$$

That is,

$$v(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t} \quad \textit{in } G_{\alpha,\sigma},\ \textit{for all } \alpha, \sigma > 0.$$

Theorem 3.1 is our key technical result. With this, we immediately obtain the asymptotic expansions for solutions of (1.16).

Theorem 3.2. *Let $u(t)$ be any Leray-Hopf weak solution of* (1.16)*. Then there exist unique $\mathcal{V}$-valued S-polynomials Q_n's, for all $n \in \mathbb{N}$, such that it holds, for any $\alpha, \sigma > 0$ and $N \geq 1$, that*

$$\Big| u(t) - \sum_{n=1}^{N} Q_n(t) e^{-\mu_n t} \Big|_{\alpha,\sigma} = \mathcal{O}\big(e^{-\mu t}\big),\ \textit{as } t \to \infty,\ \textit{for all } \mu \in (\mu_N, \mu_{N+1}). \tag{3.2}$$

That is,

$$u(t) \sim \sum_{n=1}^{\infty} Q_n(t) e^{-\mu_n t} \quad \textit{in } G_{\alpha,\sigma},\ \textit{for all } \alpha, \sigma > 0.$$

Moreover, each Q_n is related to q_n in Theorem 3.1 via relation (3.5)*, below.*

Proof. Let $T_* > 0$ be as in Theorem 1.3(iii). Set $v(t) = e^{\Omega t S} u(t)$. Then $v(t)$ is a regular solution of (1.29) on $[T_*, \infty)$. Applying Theorem 3.1 to solution $v(T_* + t)$, we have, for all $\alpha, \sigma > 0$ and $N \geq 1$, that

$$\Big| v(T_* + t) - \sum_{n=1}^{N} q_n(t) e^{-\mu_n t} \Big|_{\alpha,\sigma} = \mathcal{O}\big(e^{-\mu t}\big),\ \text{as } t \to \infty,\ \text{for all } \mu \in (\mu_N, \mu_{N+1}), \tag{3.3}$$

where all q_n's belong to $\mathcal{F}_1(\mathcal{V})$. By shifting the time variable, we obtain from (3.3) that

$$\Big| v(t) - \sum_{n=1}^{N} q_n(t - T_*) e^{-\mu_n (t - T_*)} \Big|_{\alpha,\sigma} = \mathcal{O}\big(e^{-\mu t}\big),\ \text{as } t \to \infty. \tag{3.4}$$

Let

$$Q_n(t) = e^{\mu_n T_*} e^{-\Omega t S} q_n(t - T_*). \tag{3.5}$$

Rewrite the left-hand side of (3.4) as

$$\Big| e^{\Omega t S} \Big(u(t) - \sum_{n=1}^{N} Q_n(t) e^{-\mu_n t} \Big) \Big|_{\alpha,\sigma},\ \text{which equals } \Big| u(t) - \sum_{n=1}^{N} Q_n(t) e^{-\mu_n t} \Big|_{\alpha,\sigma}$$

thanks to the isometry (1.25). Thus, we obtain (3.2). Thanks to Lemma 2.5(i), each $q_n(t - T_*)$ is a $\mathcal{V}$-valued S-polynomial, and hence, by (2.14) of Lemma 2.7, so is each $Q_n(t)$. The uniqueness of the S-polynomials Q_n's follows from Lemma 2.3. □

Our proof of Theorem 3.1 uses the Gevrey norm technique. We recall a convenient estimate in [20, Lemma 2.1] for the Gevrey norms of the the bi-linear form $B(\cdot, \cdot)$, which is a generalization of the original inequality in [19, Lemma 2.1], and also the Sobolev estimates in [9].

There exists a constant $K \geq 1$ such that for any numbers $\alpha \geq 1/2$, $\sigma \geq 0$, and any functions $v, w \in G_{\alpha+1/2,\sigma}$, one has

$$|B(v, w)|_{\alpha,\sigma} \leq K^{\alpha} |v|_{\alpha+1/2,\sigma} |w|_{\alpha+1/2,\sigma}. \tag{3.6}$$

The same estimate as (3.6) can be obtained for $B(t, v, w)$.

Lemma 3.3. *For any numbers $\alpha \ge 1/2$, $\sigma \ge 0$, any functions $v, w \in G_{\alpha+1/2,\sigma}$ and any $t \in \mathbb{R}$, one has*

$$|B(t,v,w)|_{\alpha,\sigma} \le K^\alpha |v|_{\alpha+1/2,\sigma} |w|_{\alpha+1/2,\sigma}, \tag{3.7}$$

where K is the constant in (3.6).

Proof. By the isometry (1.25) and inequality (3.6), we have

$$\begin{aligned}|B(t,v,w)|_{\alpha,\sigma} &= |B(e^{-tS}v, e^{-tS}w)|_{\alpha,\sigma}\\ &\le K^\alpha |e^{-tS}v|_{\alpha+1/2,\sigma} |e^{-tS}w|_{\alpha+1/2,\sigma} = K^\alpha |v|_{\alpha+1/2,\sigma}|w|_{\alpha+1/2,\sigma},\end{aligned}$$

which proves (3.7). □

As another preparation for the proof of Theorem 3.1, we establish the relevant estimates for the Gevrey norms of $v(t)$, when t is large.

Proposition 3.4. *Let $v^0 \in H$ and $v(t)$ be a Leray-Hopf weak solution of* (1.29). *For any $\sigma > 0$, there exist $T, D_\sigma > 0$ such that*

$$|v(t)|_{1/2,\sigma+1} \le D_\sigma e^{-t}, \quad \text{for all } t \ge T.$$

Moreover, for any $\alpha \ge 0$ there exists $D_{\alpha,\sigma} > 0$ such that

$$|v(t)|_{\alpha+1/2,\sigma} \le D_{\alpha,\sigma} e^{-t}, \quad \text{for all } \ t \ge T. \tag{3.8}$$

Proof. Thanks to the isometry (1.25), properties (1.30), (1.31), and inequality (3.7), the proof, with $\mu_1 = 1$ under the current setting, is exactly the same as in [20, Theorem 2.4] and is omitted. □

We now are ready to prove Theorem 3.1.

Proof of Theorem 3.1. This proof follows [17] and [20, 21] with necessary modifications.

Firstly, we note, by part (iii) of Theorem 1.3, that there exists $T_* > 0$ such that equation (1.29) holds in the classical sense in $\mathcal{D}(A)$ on (T_*, ∞).

Let $\sigma > 0$ be fixed. For each $N \in \mathbb{N}$, our main statement is

(H$_N$) *There exist $\mathcal{V}$-valued S-polynomials q_n's for $n = 1, 2, \dots, N$, such that*

$$\Big|v(t) - \sum_{n=1}^N q_n(t) e^{-\mu_n t}\Big|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_N+\varepsilon)t}) \text{ as } t \to \infty, \tag{3.9}$$

for all $\alpha > 0$, and some $\varepsilon = \varepsilon_{N,\alpha} > 0$. Moreover, each $v_n(t) \stackrel{\text{def}}{=} q_n(t)e^{-\mu_n t}$, for $n = 1, 2, \dots, N$, solves the equation

$$v_n'(t) + Av_n(t) + \sum_{\substack{1\le m,j\le n-1\\ \mu_m+\mu_j=\mu_n}} B_\Omega(t, v_m(t), v_j(t)) = 0, \text{ for all } t \in \mathbb{R}. \tag{3.10}$$

Claim: (H$_N$) holds true for any $N \in \mathbb{N}$.

We prove this Claim by induction in N.

First step $N = 1$. Given $\alpha \ge 1/2$. By estimate (3.8), there exist $T_0 > T_*$ and $d_0 > 0$ such that

$$|v(t)|_{\alpha+1/2,\sigma} \le d_0 e^{-\mu_1 t}, \quad \text{for all } t \ge T_0. \tag{3.11}$$

Set $w_0 = e^{\mu_1 t} v(t)$. We have, for $t \in (T_0, \infty)$, that

$$w_0' = e^{\mu_1 t}(v' + \mu_1 v) = e^{\mu_1 t}(-Av - B_\Omega(t,v,v)) + \mu_1 v),$$

hence

$$w_0' + (A-\mu_1)w_0 = H_0(t) \overset{\rm def}{=} -e^{\mu_1 t}B_\Omega(t,v(t),v(t)). \tag{3.12}$$

Estimate (3.11) and inequality (3.7) imply

$$|H_0(T_0+t)|_{\alpha,\sigma} \le e^{\mu_1(T_0+t)}K^\alpha |v(T_0+t)|^2_{\alpha+1/2,\sigma} \le M_0 e^{-\mu_1 t}, \quad \text{for all } t\ge 0, \tag{3.13}$$

where $M_0 = K^\alpha d_0^2 e^{-\mu_1 T_0}$.

For $k\in\mathbb{N}$, applying the projection R_{Λ_k} to equation (3.12) gives

$$(R_{\Lambda_k}w_0)' + (\Lambda_k-\mu_1)R_{\Lambda_k}w_0 = R_{\Lambda_k}H_0(t). \tag{3.14}$$

We apply Lemma 2.9 to equation (3.14) in the space $X=R_{\Lambda_k}H$ with norm $\|\cdot\|_X = |\cdot|_{\alpha,\sigma}$, solution $y(t)=R_{\Lambda_k}w_0(T_0+t)$, S-polynomial $p(t)\equiv 0$, constant $\beta=\Lambda_k-\mu_1\ge 0$, function $g(t)=R_{\Lambda_k}H_0(T_0+t)$ and numbers $M=M_0$, $\delta=\mu_1$ in (2.22).

When $k=1$, we have $\beta=0$, then by Lemma 2.9(ii), it follows that

$$|R_{\Lambda_1}w_0(T_0+t)-\xi_1|_{\alpha,\sigma} = \mathcal{O}(e^{-\mu_1 t}), \tag{3.15}$$

where

$$\xi_1 = R_{\Lambda_1}w_0(T_0) + \int_0^\infty e^{\mu_1\tau}R_{\Lambda_1}H_0(T_0+\tau)d\tau,$$

which exists and belongs to $R_{\Lambda_1}H$.

When $k\ge 2$, we have $\beta\ge\mu_2-\mu_1>0$, and it follows Lemma 2.9(i) that $q(t)$ defined by (2.23) is 0, and, by (2.25), one has

$$|R_{\Lambda_k}w_0(T_0+t)|^2_{\alpha,\sigma} \le 2e^{-2(\Lambda_k-\mu_1)t}|R_{\Lambda_k}w_0(T_0)|^2_{\alpha,\sigma} + 2t\int_0^t e^{-2(\Lambda_k-\mu_1)(t-\tau)}|R_{\Lambda_k}H_0(T_0+\tau)|^2_{\alpha,\sigma}d\tau$$

$$\le 2e^{-2(\mu_2-\mu_1)t}|R_{\Lambda_k}w_0(T_0)|^2_{\alpha,\sigma} + 2t\int_0^t e^{-2(\mu_2-\mu_1)(t-\tau)}|R_{\Lambda_k}H_0(T_0+\tau)|^2_{\alpha,\sigma}d\tau.$$

Summing up this inequality in k gives

$$\begin{aligned}
|(\mathrm{Id}-R_{\Lambda_1})w_0(T_0+t)|^2_{\alpha,\sigma} &= \sum_{k=2}^\infty |R_{\Lambda_k}w_0(T_0+t)|^2_{\alpha,\sigma}\\
&\le 2e^{-2(\mu_2-\mu_1)t}\sum_{k=2}^\infty |R_{\Lambda_k}w_0(T_0)|^2_{\alpha,\sigma} + 2t\int_0^t e^{-2(\mu_2-\mu_1)(t-\tau)}\sum_{k=2}^\infty |R_{\Lambda_k}H_0(T_0+\tau)|^2_{\alpha,\sigma}d\tau\\
&\le 2e^{-2(\mu_2-\mu_1)t}|w_0(T_0)|^2_{\alpha,\sigma} + 2te^{-2(\mu_2-\mu_1)t}\int_0^t e^{2(\mu_2-\mu_1)\tau}|H_0(T_0+\tau)|^2_{\alpha,\sigma}d\tau.
\end{aligned}$$

Using (3.13), we obtain

$$|(\mathrm{Id}-R_{\Lambda_1})w_0(T_0+t)|^2_{\alpha,\sigma} \le 2e^{-2(\mu_2-\mu_1)t}\Big(|w_0(T_0)|^2_{\alpha,\sigma} + 2M_0 t\int_0^t e^{2(\mu_2-2\mu_1)\tau}d\tau\Big).$$

We simply use the fact $\mu_2\le 2\mu_1$ in the last integral, and obtain

$$|(\mathrm{Id}-R_{\Lambda_1})w_0(T_0+t)|^2_{\alpha,\sigma} \le 2e^{-2(\mu_2-\mu_1)t}\big(|w_0(T_0)|^2_{\alpha,\sigma}+2M_0t^2\big). \tag{3.16}$$

Combining (3.15), (3.16) and the fact $\mu_2-\mu_1\le\mu_1$, gives

$$|w_0(t)-\xi_1|_{\alpha,\sigma} \le |R_{\Lambda_1}w_0(t)-\xi_1|_{\alpha,\sigma} + |(\mathrm{Id}-R_{\Lambda_1})w_0(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\varepsilon t}), \tag{3.17}$$

for any number ε such that $0<\varepsilon<\mu_2-\mu_1$.

Define

$$q_1(t) \equiv \xi_1. \tag{3.18}$$

Multiplying (3.17) by $e^{-\mu_1 t}$ yields (3.9) for $N = 1$. Also, since $\xi_1 \in R_{\Lambda_1} H$, it is clear that $v_1(t) = \xi_1 e^{-\Lambda_1 t}$ satisfies $v_1'(t) + Av_1(t) = 0$ on $\mathbb{R}$. Hence, v_1 satisfies (3.10) with $n = 1$, because the sum of the bi-linear terms in (3.10) is void. Note that q_1 does not depend on α. Therefore, the statement $(\mathbf{H}_N)$ holds true for $N = 1$.

Induction step. Let $N \ge 1$ and assume the statement $(\mathbf{H}_N)$ holds true. Let q_n, for $n = 1, 2, \ldots, N$, be the $\mathcal{V}$-valued S-polynomials in $(\mathbf{H}_N)$. There exists $\Lambda \in \sigma(A)$ such that

$$q_n \in \mathcal{F}_1(P_\Lambda H), \text{ for all } 1 \le n \le N. \tag{3.19}$$

Let $v_n(t) = q_n(t)e^{-\mu_n t}$, denote $s_N(t) = \sum_{n=1}^N v_n(t)$ and $r_N(t) = v(t) - s_N(t)$. Let $\alpha \ge 1/2$.

(a) By the definition of v_n, we have, for $n \ge 2$,

$$|v_n(t)|_{\alpha+1/2,\sigma} = \mathcal{O}(e^{-(\mu_n-\delta)t}) \quad \forall \delta > 0, \tag{3.20}$$

and, thanks to (3.18),

$$|v_1(t)|_{\alpha+1/2,\sigma} = \mathcal{O}(e^{-\mu_1 t}).$$

The last two properties imply

$$|s_N(T+t)|_{\alpha+1/2,\sigma} = \mathcal{O}(e^{-\mu_1 t}). \tag{3.21}$$

By the induction hypothesis $(\mathbf{H}_N)$ applied to $\alpha + 1/2$, there exists $\varepsilon > 0$ such that

$$|r_N(t)|_{\alpha+1/2,\sigma} = \mathcal{O}(e^{-(\mu_N+\varepsilon)t}). \tag{3.22}$$

We derive a differential equation for $r_N(t)$, for $t > T_*$. We calculate from (1.29) and (3.10) for $n = 1, 2, \ldots, N$ that

$$\begin{aligned} r_N' &= v' - \sum_{n=1}^N v_n' = -Av - B_\Omega(t,v,v) - \sum_{n=1}^N \Big\{ -Av_n - \sum_{\substack{1\le m,j\le n-1\\ \mu_m+\mu_j=\mu_n}} B_\Omega(t,v_m,v_j) \Big\} \\ &= -Ar_N - B_\Omega(t,r_N(t),v(t)) - B_\Omega(t,s_N(t),r_N(t)) - B_\Omega(t,s_N(t),s_N(t)) \\ &\quad + \sum_{\mu_m+\mu_j\le\mu_N} B_\Omega(t,v_m,v_j). \end{aligned}$$

We manipulate the last two terms as

$$\begin{aligned} &- B_\Omega(t,s_N(t),s_N(t)) + \sum_{\mu_m+\mu_j\le\mu_N} B_\Omega(t,v_m,v_j) = - \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j\ge\mu_{N+1}}} B_\Omega(t,v_m(t),v_j(t)) \\ &= - \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} B_\Omega(t,v_m,v_j) - \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j\ge\mu_{N+2}}} B_\Omega(t,v_m(t),v_j(t)). \end{aligned}$$

Thus, we obtain

$$r_N'(t) + Ar_N(t) + \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} B_\Omega(t,v_m(t),v_j(t)) = h_N, \text{ for } t > T_*, \tag{3.23}$$

where

$$h_N(t) = -B_\Omega(t, r_N(t), v(t)) - B_\Omega(t, s_N(t), r_N(t)) - \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j\ge\mu_{N+2}}} B_\Omega(t, v_m(t), v_j(t)). \tag{3.24}$$

(b) We estimate each term on the right-hand side of (3.24). Note from (3.11), (3.22), (3.20), (3.21) and (3.7) that

$$|B_\Omega(t, r_N(t), v(t))|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_N+\mu_1+\varepsilon)t}), \tag{3.25}$$

$$|B_\Omega(t, s_N(t), r_N(t))|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_N+\mu_1+\varepsilon)t}), \tag{3.26}$$

and for $1 \le m, j \le N$ with $\mu_m + \mu_j \ge \mu_{N+2}$,

$$|B_\Omega(t, v_m(t), v_j(t))|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_m+\mu_j-2\delta)t}) = \mathcal{O}(e^{-(\mu_{N+2}-2\delta)t}), \forall \delta > 0. \tag{3.27}$$

Since $\mu_N + \mu_1 \ge \mu_{N+1}$, $2\mu_N \ge \mu_{N+1}$ and $\mu_{N+2} > \mu_{N+1}$, by taking δ sufficiently small in (3.27), we have from (3.24), (3.25), (3.26) and (3.27) that

$$|h_N(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-(\mu_{N+1}+\delta_N)t}), \quad \text{for some } \delta_N \in (0, \mu_{N+2} - \mu_{N+1}). \tag{3.28}$$

(c) Define $w_N(t) = e^{\mu_{N+1}t} r_N(t)$, and $w_{N,k}(t) = R_{\Lambda_k} w_N(t)$, for $k \in \mathbb{N}$. We have from (3.23) that

$$\frac{d}{dt} w_{N,k} + (\Lambda_k - \mu_{N+1}) w_{N,k} = -\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(t, q_m, q_j) + R_{\Lambda_k} H_N(t), \tag{3.29}$$

where $H_N(t) = e^{\mu_{N+1}t} h_N(t)$. By (3.19) and (2.16),

$$B_\Omega(t, q_m(t), q_j(t)) \in \mathcal{F}_1(P_{4\Lambda} H), \tag{3.30}$$

which implies that the finite sum

$$\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} B_\Omega(t, q_m(t), q_j(t)) \in \mathcal{F}_1(P_{4\Lambda} H).$$

Consequently,

$$\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(t, q_m(t), q_j(t)) \in \mathcal{F}_1(R_{\Lambda_k} H). \tag{3.31}$$

By the first property in Lemma 2.5(i),

$$\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(T + t, q_m(T + t), q_j(T + t)) \in \mathcal{F}_1(R_{\Lambda_k} H), \quad \text{for all } T \in \mathbb{R}.$$

By (3.28), $|H_N(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t})$. Then there exist $T_N > T_*$ and $M_N > 0$ such that

$$|H_N(T_N + t)|_{\alpha,\sigma} \le M_N e^{-\delta_N t}, \quad \text{for all } t \ge 0. \tag{3.32}$$

We will apply Lemma 2.9 again to equation (3.29) in the space $X = R_{\Lambda_k} H$ with norm $\|\cdot\|_X = |\cdot|_{\alpha,\sigma}$, solution $y(t) = w_{N,k}(T_N + t)$, constant $\beta = \Lambda_k - \mu_{N+1}$, S-polynomial

$$p(t) = -\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(T_N + t, q_m(T_N + t), q_j(T_N + t)),$$

function $g(t) = R_{\Lambda_k} H_N(T_N + t)$, numbers $M = M_N$ and $\delta = \delta_N$ in (3.32).

We consider three cases.

Case $\Lambda_k = \mu_{N+1}$. Then $\beta = 0$ in (3.29). Let

$$\xi_{N+1} \stackrel{\rm def}{=} R_{\mu_{N+1}} r_N(T_N) + \int_0^\infty e^{\mu_{N+1}\tau} R_{\mu_{N+1}} H_N(T_N+\tau)\, d\tau,$$

which exists and belongs to $R_{\mu_{N+1}}H$. Define

$$p_{N+1,k}(t) = \xi_{N+1} - \int_0^t \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\mu_{N+1}} B_\Omega(T_N+\tau, q_m(T_N+\tau), q_j(T_N+\tau))d\tau. \tag{3.33}$$

Case $\Lambda_k \le \mu_N$. Then $\beta < 0$ in (3.29). Note, by (3.22), that

$$e^{\beta t}|y(t)|_{\alpha,\sigma} = e^{\Lambda_k t}|R_{\Lambda_k} r_N(T_N+t)|_{\alpha,\sigma} \le e^{\mu_N t}|R_{\Lambda_k} r_N(T_N+t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\varepsilon t}).$$

Hence condition (2.26) is met. Define

$$\begin{aligned} p_{N+1,k}(t) = e^{-(\Lambda_k-\mu_{N+1})t} \int_t^\infty e^{(\Lambda_k-\mu_{N+1})\tau} \\ \cdot \Big(\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(T_N+\tau, q_m(T_N+\tau), q_j(T_N+\tau))\Big) d\tau. \end{aligned} \tag{3.34}$$

In the above two cases of Λ_k, by applying Lemma 2.9(ii), we obtain $p_{N+1,k} \in \mathcal{F}_1(R_{\Lambda_k}H)$ that satisfies

$$|w_{N,k}(T_N+t) - p_{N+1,k}(t)|^2_{\alpha,\sigma} = \mathcal{O}(e^{-2\delta_N t}). \tag{3.35}$$

Case $\Lambda_k \ge \mu_{N+2}$. Then $\beta > 0$ in (3.29). Define

$$\begin{aligned} p_{N+1,k}(t) = -e^{-(\Lambda_k-\mu_{N+1})t} \int_{-\infty}^t e^{(\Lambda_k-\mu_{N+1})\tau} \\ \cdot \Big(\sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(T_N+\tau, q_m(T_N+\tau), q_j(T_N+\tau))\Big) d\tau. \end{aligned} \tag{3.36}$$

Applying Lemma 2.9(i), we obtain $p_{N+1,k} \in \mathcal{F}_1(R_{\Lambda_k}H)$, and, by (2.25),

$$\begin{aligned} |w_{N,k}(T_N+t) - p_{N+1,k}(t)|^2_{\alpha,\sigma} \le 2e^{-2(\Lambda_k-\mu_{N+1})t}|w_{N,k}(T_N) - p_{N+1,k}(0)|^2_{\alpha,\sigma} \\ + 2t\int_0^t e^{-2(\mu_{N+2}-\mu_{N+1})(t-\tau)}|R_{\Lambda_k} H_N(T_N+\tau)|^2_{\alpha,\sigma} d\tau. \end{aligned} \tag{3.37}$$

Denote $z_{N,k}(t) = w_{N,k}(T_N+t) - p_{N+1,k}(t)$, which is the remainder on the left-hand side of (3.35) and (3.37).

On the one hand, because there are only finitely many k's with $\Lambda_k \le \mu_{N+1}$, it follows from (3.35) that

$$\sum_{\Lambda_k\le\mu_{N+1}} |z_{N,k}(t)|^2_{\alpha,\sigma} = \mathcal{O}(e^{-2\delta_N t}). \tag{3.38}$$

On the other hand, summing inequality (3.37) in k for which $\Lambda_k \geq \mu_{N+2}$, we obtain

$$\sum_{\Lambda_k \geq \mu_{N+2}} |z_{N,k}(t)|^2_{\alpha,\sigma} \leq 4e^{-2(\mu_{N+2}-\mu_{N+1})t} \sum_{\Lambda_k \geq \mu_{N+2}} |w_{N,k}(T_N) - p_{N+1,k}(0)|^2_{\alpha,\sigma}$$
$$+ 2t \int_0^t e^{-2(\mu_{N+2}-\mu_{N+1})(t-\tau)} \sum_{\Lambda_k \geq \mu_{N+2}} |R_{\Lambda_k} H_N(T_N + \tau)|^2_{\alpha,\sigma} d\tau. \quad (3.39)$$

For the terms on the right-hand side of (3.39), we have

$$\sum_{\Lambda_k \geq \mu_{N+2}} |w_{N,k}(T_N) - p_{N+1,k}(0)|^2_{\alpha,\sigma} \leq 2 \sum_{\Lambda_k \geq \mu_{N+2}} |R_{\Lambda_k} w_N(T_N)|^2_{\alpha,\sigma} + 2 \sum_{\Lambda_k \geq \mu_{N+2}} |p_{N+1,k}(0)|^2_{\alpha,\sigma}$$
$$\leq 2|w_N(T_N)|^2_{\alpha,\sigma} + 2 \sum_{\Lambda_k \geq \mu_{N+2}} |p_{N+1,k}(0)|^2_{\alpha,\sigma}.$$

It follows from (3.31) and (3.36) that $p_{N+1,k} \equiv 0$, for $\Lambda_k > \max\{4\Lambda, \mu_{N+1}\}$. Thus,

$$\sum_{\Lambda_k \geq \mu_{N+2}} |p_{N+1,k}(0)|^2_{\alpha,\sigma} \text{ is a finite sum, and hence is finite.}$$

Also, the last integral in (3.39) has

$$\sum_{\Lambda_k \geq \mu_{N+2}} |R_{\Lambda_k} H_N(T_N + \tau)|^2_{\alpha,\sigma} \leq |H_N(T_N + \tau)|^2_{\alpha,\sigma} \leq M_N^2 e^{-2\delta_N \tau}.$$

Therefore, there exists $C_0 > 0$ such that

$$\sum_{\Lambda_k \geq \mu_{N+2}} |z_{N,k}(t)|^2_{\alpha,\sigma} \leq 4C_0 e^{-2(\mu_{N+2}-\mu_{N+1})t} + 2M_N^2 t e^{-2(\mu_{N+2}-\mu_{N+1})t} \int_0^t e^{2(\mu_{N+2}-\mu_{N+1}-\delta_N)\tau} d\tau.$$

Recall that $\delta_N < \mu_{N+2} - \mu_{N+1}$ in (3.32). Calculating the last integral explicitly, we easily find

$$\sum_{\Lambda_k \geq \mu_{N+2}} |z_{N,k}(t)|^2_{\alpha,\sigma} \leq 4C_0 e^{-2(\mu_{N+2}-\mu_{N+1})t} + \frac{M_N^2 t e^{-2\delta_N t}}{\mu_{N+2} - \mu_{N+1} - \delta_N} \leq C_N e^{-\delta_N t}, \quad (3.40)$$

for some constant $C_N > 0$.

Combining (3.38) and (3.40) gives

$$\sum_{k=1}^{\infty} |z_{N,k}(t)|^2_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t}). \quad (3.41)$$

(d) Let Λ_* be the smallest Λ_n such that $\Lambda_n \geq \max\{4\Lambda, \mu_{N+1}\}$. By (3.30) and formulas (3.33), (3.34), (3.36), we have

$$p_{N+1,k} \in \mathcal{F}_1(P_{\Lambda_k} H) \subset \mathcal{F}_1(P_{\Lambda_*} H), \text{ for } \Lambda_k \leq \Lambda_*, \text{ and } p_{N+1,k} = 0, \text{ for } \Lambda_k > \Lambda_*. \quad (3.42)$$

Define, for $t \in \mathbb{R}$,

$$q_{N+1}(t) = \sum_{k=1}^{\infty} p_{N+1,k}(t - T_N). \quad (3.43)$$

Thanks to (3.42), the sum in (3.43) is only a finite sum, and $q_{N+1} \in \mathcal{F}_1(P_{\Lambda_*} H)$.

Obviously,

$$R_{\Lambda_k} q_{N+1}(t + T_N) = p_{N+1,k}(t), \quad (3.44)$$

hence, $z_{N,k}(t) = R_{\Lambda_k}\big(w_N(t+T_N) - q_{N+1}(t+T_N)\big)$. It follows from (3.41) that

$$|w_N(t+T_N) - q_{N+1}(t+T_N)|^2_{\alpha,\sigma} = \sum_{k=1}^\infty |z_{N,k}(t)|^2_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t}),$$

which implies that

$$|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\delta_N t/2}).$$

Multiplying this equation by $e^{-\mu_{N+1}t}$, we obtain the desired statement (3.9) for $N+1$.

(e) It remains to prove the ODE (3.10), for $n = N+1$. By (2.24) of Lemma 2.9, we have, for each k, that

$$p'_{N+1,k}(t) + (\Lambda_k - \mu_{N+1})p_{N+1,k}(t) + \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} R_{\Lambda_k} B_\Omega(T_N+t, q_m(T_N+t), q_j(T_N+t)) = 0.$$

From this and (3.44), we deduce

$$(R_{\Lambda_k} q_{N+1}(t))' + (\Lambda_k - \mu_{N+1})R_{\Lambda_k} q_{N+1}(t) + R_{\Lambda_k} \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} B_\Omega(t, q_m(t), q_j(t)) = 0.$$

Multiplying this equation by $e^{-\mu_{N+1}t}$ yields

$$(R_{\Lambda_k} v_{N+1}(t))' + A(R_{\Lambda_k} v_{N+1}(t)) + R_{\Lambda_k} \sum_{\substack{1\le m,j\le N\\ \mu_m+\mu_j=\mu_{N+1}}} B_\Omega(t, v_m(t), v_j(t)) = 0,$$

where $v_{N+1}(t) = q_{N+1}(t)e^{-\mu_{N+1}t}$. Note in this equation that $v_{N+1}(t)$ and $B_\Omega(t, v_m(t), v_j(t))$ all belong to $P_{\Lambda_*}H$. Then summing up the equation in k, for which $\Lambda_k \le \Lambda_*$, yields that the ODE system (3.10) holds for $n = N+1$. According to Remark 2.4, the S-polynomial q_{N+1} is independent of α. Therefore, the statement $(\mathbf{H}_{N+1})$ is proved.

By the induction principle, the statement $(\mathbf{H}_N)$ holds true for all N, i.e., the Claim is proved.

Now, we note that at step $n = N+1$, the q_n's, for $n = 1, 2, \ldots, N$, are those from the step $n = N$, and are used in the construction of q_{N+1}. Therefore, for each $\sigma > 0$, such recursive construction gives the existence of the S-polynomials q_n's, for all $n \in \mathbb{N}$. By Remark 2.4 again, all these q_n's are, in fact, independent of σ. Therefore, (3.9) holds true for all N, α, σ. Then estimate (3.1) follows thanks to (2.7). Finally, by Lemma 2.3, the S-polynomials q_n's are unique. The proof is complete. □

Remark 3.5. The statement and proof of Theorem 3.1 can be presented simply with $G_{0,\sigma}$ for all $\sigma \ge 0$. Nonetheless, general calculations in the above proof for $G_{\alpha,\sigma}$ are flexible and can be applied to cover the case when a non-potential force is included in the NSE and has limited regularity in $G_{\alpha,\sigma}$ for a fixed σ, see [2, 3, 21].

Remark 3.6. It is not known whether the polynomials Q_n's in Theorem 3.2 have any limits as $|\Omega| \to \infty$. However, such a question can be answered for some special solutions, see Remark 5.5 below.

4. The case of non-zero spatial average solutions

In this section, we establish the asymptotic expansions for the solutions with non-zero spatial averages.

Assumption 4.1. *Throughout this section,* $\mathbf{u}(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(\mathbb{R}^3 \times (0,\infty)) \cap C(\mathbb{R}^3 \times [0,\infty))$ *and* $p(\mathbf{x},t) \in C^1_{\mathbf{x}}(\mathbb{R}^3 \times (0,\infty))$ *are* $\mathbf{L}$*-periodic functions that form a solution* $(\mathbf{u},p)$ *of the NSE* (1.1) *and* (1.2).

Note that any $\mathbf{L}$-periodic function on $\mathbb{R}^3$ can be considered as a function on the flat torus $\mathbb{T}_{\mathbf{L}}$. In particular, $u(t) = \mathbf{u}(\cdot,t)$ and $p(t) = p(\cdot,t)$ can be considered as functions on $\mathbb{T}_{\mathbf{L}}$.

Suppose $\mathbf{f}$ is a $\mathbf{L}$-periodic vector field on $\mathbb{R}^3$ and $\mathbf{f} \in \mathcal{D}(A^\alpha e^{\sigma A^{1/2}})$, for some $\alpha, \sigma \geq 0$. Let $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}+\mathbf{X}_0)$, for some fixed $\mathbf{X}_0 \in \mathbb{R}^3$, then $\mathbf{g} \in H$. Let $\mathbf{f}(\mathbf{x}) = \sum' \widehat{\mathbf{f}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}$, then

$$\mathbf{g}(\mathbf{x}) = \sum{}' \widehat{\mathbf{g}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \text{ where } \widehat{\mathbf{g}}_{\mathbf{k}} = \widehat{\mathbf{f}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{X}_0}.$$

It follows that $|\widehat{\mathbf{g}}_{\mathbf{k}}| = |\widehat{\mathbf{f}}_{\mathbf{k}}|$, and consequently,

$$|\mathbf{f}(\cdot+\mathbf{X}_0)|_{\alpha,\sigma} = |\mathbf{g}(\cdot)|_{\alpha,\sigma} = |\mathbf{f}(\cdot)|_{\alpha,\sigma}. \tag{4.1}$$

Returning to the solution $(\mathbf{u},p)$, denote, for $t \geq 0$,

$$\mathbf{U}(t) = \frac{1}{L_1L_2L_3}\int_{\mathbb{T}_{\mathbf{L}}} \mathbf{u}(\mathbf{x},t)d\mathbf{x}.$$

Integrating equation (1.1) over the domain $\mathbb{T}_{\mathbf{L}}$ gives

$$\mathbf{U}'(t) + \Omega J\mathbf{U}(t) = 0, \quad t > 0.$$

Hence,

$$\mathbf{U}(t) = e^{-\Omega tJ}\mathbf{U}(0) = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{U}(0). \tag{4.2}$$

Theorem 4.2. *There exist* $\mathcal{V}$*-valued SS-polynomials* $\mathcal{Q}_n(t)$*'s, for all* $n \in \mathbb{N}$*, such that*

$$u(t) - \mathbf{U}(t) \sim \sum_{n=1}^\infty \mathcal{Q}_n(t)e^{-\mu_n t} \text{ in } G_{\alpha,\sigma}, \text{ for all } \alpha,\sigma > 0. \tag{4.3}$$

Proof. For $t \geq 0$, define $\mathbf{V}(t) = \int_0^t \mathbf{U}(\tau)d\tau$, which, by (4.2), is

$$\mathbf{V}(t) = \frac{1}{\Omega}\begin{pmatrix} \sin(\Omega t) & 1-\cos(\Omega t) & 0 \\ \cos(\Omega t) - 1 & \sin(\Omega t) & 0 \\ 0 & 0 & \Omega t \end{pmatrix} \mathbf{U}(0). \tag{4.4}$$

We use the following hyper-Galilean transformation [6]:

$$\mathbf{w}(\mathbf{x},t) = \mathbf{u}(\mathbf{x}+\mathbf{V}(t),t) - \mathbf{U}(t), \quad \vartheta(\mathbf{x},t) \mapsto p(\mathbf{x}+\mathbf{V}(t),t). \tag{4.5}$$

By simple calculations, one can verify

$$\mathbf{w}_t - \Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \Omega J\mathbf{w} = -\nabla\vartheta \text{ and } \operatorname{div}\mathbf{w} = 0,$$

that is, $(\mathbf{w},\vartheta)$ is also a classical solution of the system (1.1) and (1.2) on $\mathbb{R}^3 \times (0,\infty)$.

Note that $(\mathbf{w},\vartheta)$ is $\mathbf{L}$-periodic, and $\mathbf{w}(\cdot,t)$ has zero average for each $t \geq 0$. Applying Theorem 3.2 to the solution $w(t) = \mathbf{w}(\cdot,t)$ we obtain

$$w(t) \sim \sum_{n=1}^\infty Q_n(t)e^{-\mu_n t} \text{ in } G_{\alpha,\sigma}, \text{ for all } \alpha,\sigma > 0, \tag{4.6}$$

where $Q_n(t)$'s are $\mathcal{V}$-valued S-polynomials.

Assume each $Q_n(t)$ is a mapping $\mathbf{x} \mapsto \mathbf{Q}_n(\mathbf{x},t)$. Note that each $Q_n(t)$ belongs to $\mathcal{V}$, hence $\mathbf{Q}_n(\mathbf{x},t)$, as a function of $\mathbf{x} \in \mathbb{R}^3$, is $\mathbf{L}$-periodic. Let $N \in \mathbb{N}$, $\alpha, \sigma > 0$ and $\mu \in (\mu_N, \mu_{N+1})$. We have from (4.6) and (2.7) that

$$\Big|\mathbf{u}(\mathbf{x}+\mathbf{V}(t),t) - \mathbf{U}(t) - \sum_{n=1}^{N} \mathbf{Q}_n(\mathbf{x},t)e^{-\mu_n t}\Big|_{\alpha,\sigma} = \mathcal{O}(e^{-\mu t}).$$

This and (4.1) imply

$$\Big|\mathbf{u}(\mathbf{x},t) - \mathbf{U}(t) - \sum_{n=1}^{N} \mathbf{Q}_n(\mathbf{x}-\mathbf{V}(t),t)e^{-\mu_n t}\Big|_{\alpha,\sigma} = \mathcal{O}(e^{-\mu t}),$$

which means

$$\Big|u(t) - \mathbf{U}(t) - \sum_{n=1}^{N} \mathcal{Q}_n(t)e^{-\mu_n t}\Big|_{\alpha,\sigma} = \mathcal{O}(e^{-\mu t}), \tag{4.7}$$

where $\mathcal{Q}_n(t) = \mathbf{Q}_n(\cdot - \mathbf{V}(t), t)$ for $n \in \mathbb{N}$. It remains to prove that $\mathcal{Q}_n \in \mathcal{F}_2(\mathcal{V})$, for all $n \in \mathbb{N}$. Suppose

$$\mathbf{Q}_n(\mathbf{x},t) = \sum_{\text{finitely many } \mathbf{k}}{}' \widehat{\mathbf{Q}}_{n,\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \text{ with } \widehat{\mathbf{Q}}_{n,\mathbf{k}}(t) \in \mathcal{F}_1(X_{\mathbf{k}}).$$

Then

$$\mathcal{Q}_n(t) = \sum_{\text{finitely many } \mathbf{k}}{}' \widehat{\mathbf{Q}}_{n,\mathbf{k}}(t)e^{-i\check{\mathbf{k}}\cdot\mathbf{V}(t)}e^{i\check{\mathbf{k}}\cdot\mathbf{x}} = \sum_{\text{finitely many } \mathbf{k}}{}' \widehat{\mathcal{Q}}_{n,\mathbf{k}}(t)e^{i\check{\mathbf{k}}\cdot\mathbf{x}},$$

where $\widehat{\mathcal{Q}}_{n,\mathbf{k}}(t) = \widehat{\mathbf{Q}}_{n,\mathbf{k}}(t)\big[\cos(\check{\mathbf{k}}\cdot\mathbf{V}(t)) - i\sin(\check{\mathbf{k}}\cdot\mathbf{V}(t))\big]$.

Using the formula of $\mathbf{V}(t)$ in (4.4), one can see that

$$\check{\mathbf{k}}\cdot\mathbf{V}(t) = r_1\cos(\Omega t) + r_2\sin(\Omega t) + r_3 t + r_4,$$

for some numbers $r_2, r_2, r_3, r_4 \in \mathbb{R}$, that depend on $\check{\mathbf{k}}$.

By using the product to sum formulas between trigonometric functions, we have

$$\widehat{\mathbf{Q}}_{n,\mathbf{k}}(t)\cos(\check{\mathbf{k}}\cdot\mathbf{V}(t)), \widehat{\mathbf{Q}}_{n,\mathbf{k}}(t)\sin(\check{\mathbf{k}}\cdot\mathbf{V}(t)) \in \mathcal{F}_2(X_{\mathbf{k}}).$$

Therefore, $\widehat{\mathcal{Q}}_{n,\mathbf{k}}(t) \in \mathcal{F}_2(X_{\mathbf{k}})$, and, by Lemma 2.6, $\mathcal{Q}_n(t) \in \mathcal{F}_2(\mathcal{V})$. With this fact, we obtain the expansion (4.3) from estimate (4.7). □

Remark 4.3. We can roughly rewrite (4.3) as an asymptotic expansion of $u(t)$ as

$$u(t) \sim \mathbf{U}(t) + \sum_{n=1}^{\infty} \mathcal{Q}_n(t)e^{-\mu_n t}.$$

This shows that $\mathbf{U}(t)$ is the leading order term in the asymptotic approximation, as $t \to \infty$, i.e., the $\mathcal{O}(1)$ term, is explicitly determined by (4.2), and is responsible for the non-zero average of $u(t)$. The following order terms in the approximation are all exponentially decaying and are with zero spatial averages.

5. Some special solutions

We present next some special solutions of (1.1) and (1.2). They are inspired by examples in [7, 8, 16] for the case without rotation.

Let $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ with $\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, \widetilde{k}_3)$ be fixed. For any $m \in \mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$, we note from (1.5), (1.6) and (1.23) that

$$\check{m\mathbf{k}} = m\check{\mathbf{k}}, \quad \widetilde{m\mathbf{k}} = \mathrm{sgn}(m)\widetilde{\mathbf{k}}, \text{ and } J_{m\mathbf{k}} = \mathrm{sgn(m)J_{\mathbf{k}}}. \tag{5.1}$$

We define $V_{\mathbf{k}}$ to be the space of all $u \in V$ such that

$$u = \sum_{m\in\mathbb{Z}_*} \widehat{\mathbf{u}}_{m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{5.2}$$

By (1.12), one immediately sees that

$$(u \cdot \nabla)v = 0, \text{ for all } u, v \in V_{\mathbf{k}}. \tag{5.3}$$

Let $u \in V_{\mathbf{k}}$ as in (5.2) and $t \in \mathbb{R}$. We have, by (1.22), (5.2) and (5.1),

$$e^{tS}u = \sum_{m\in\mathbb{Z}_*} E_{\mathbf{k}}(\widetilde{k}_3 t)\widehat{\mathbf{u}}_{m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}.$$

Hence

$$e^{tS}u \in V_{\mathbf{k}}. \tag{5.4}$$

Combining (5.3) and (5.4) yields

$$[(e^{tS}u) \cdot \nabla](e^{tS}v) = 0, \text{ for all } u, v \in V_{\mathbf{k}} \text{ and } t \in \mathbb{R}. \tag{5.5}$$

Theorem 5.1. *Let $\mathbf{k} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ be fixed and $u_0 \in V_{\mathbf{k}}$, then problem (1.16), with initial condition $u(0) = u_0$, has a unique global regular solution*

$$u(t) = e^{-tA}e^{-\Omega tS}u_0, \text{ for } t \geq 0. \tag{5.6}$$

Moreover, $u(t)$ solves the linear equation

$$u_t + Au + \Omega Su = 0, \text{ for } t > 0. \tag{5.7}$$

Proof. Let $u_0 = \sum_{m\in\mathbb{Z}_*} \widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}} \in V_{\mathbf{k}}$. Set $v(t) = e^{-tA}u_0$. First, we have

$$v_t + Av = 0, \text{ for all } t > 0. \tag{5.8}$$

Note that $u(t)$ defined by (5.6) equals $e^{-\Omega tS}v(t)$. Then $u(0) = u_0$ and, by (5.8), $u(t)$ solves (5.7).

Clearly, $v(t) \in V_{\mathbf{k}}$, for all $t \geq 0$, hence, by (5.5),

$$(u(t) \cdot \nabla)u(t) = 0, \tag{5.9}$$

which gives $B(u(t), u(t)) = 0$. The last fact and (5.7) imply that $u(t)$ also solves (1.16).

Since $u(t)$ is a global, regular solution, it is unique. □

Remark 5.2. We find some special solutions for which the helicity $\mathcal{H}(t) \stackrel{\text{def}}{=} \langle \nabla \times u(t), u(t)\rangle$ vanishes for all $t \geq 0$. (See [23, 24] for the physics of helicity, and [7, 8] for its analysis in the case of non-rotating fluids.) We consider the solution in Theorem 5.1, which can be written explicitly as

$$u(t) = \sum_{m\in\mathbb{Z}_*} e^{-m^2|\check{\mathbf{k}}|^2 t}\Big\{\cos(\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} - \sin(\widetilde{k}_3\Omega t)\widetilde{\mathbf{k}} \times \widehat{\mathbf{u}}_{0,m\mathbf{k}}\Big\} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{5.10}$$

Then the vorticity is

$$\nabla \times u(t) = \sum_{m\in\mathbb{Z}_*} e^{-m^2|\check{\mathbf{k}}|^2 t} im|\check{\mathbf{k}}| \Big\{ \cos(\widetilde{k}_3\Omega t)\widetilde{\mathbf{k}} \times \widehat{\mathbf{u}}_{0,m\mathbf{k}} + \sin(\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} \Big\} e^{im\check{\mathbf{k}}\cdot\mathbf{x}},$$

and the helicity is

$$\begin{aligned}\mathcal{H}(t) &= L_1L_2L_3 \sum_{m\in\mathbb{Z}_*} e^{-2m^2|\check{\mathbf{k}}|^2 t} im|\check{\mathbf{k}}| \Big\{ \cos^2(\widetilde{k}_3\Omega t) J_{\mathbf{k}}\widehat{\mathbf{u}}_{0,m\mathbf{k}} \cdot \overline{\widehat{\mathbf{u}}}_{0,m\mathbf{k}} - \sin^2(\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} \cdot J_{\mathbf{k}}\overline{\widehat{\mathbf{u}}}_{0,m\mathbf{k}} \\ &\quad - \cos(\widetilde{k}_3\Omega t)\sin(\widetilde{k}_3\Omega t) \Big[\widehat{\mathbf{u}}_{0,m\mathbf{k}} \cdot \overline{\widehat{\mathbf{u}}}_{0,m\mathbf{k}} - J_{\mathbf{k}}\widehat{\mathbf{u}}_{0,m\mathbf{k}} \cdot J_{\mathbf{k}}\overline{\widehat{\mathbf{u}}}_{0,m\mathbf{k}} \Big] \Big\} \\ &= L_1L_2L_3 \sum_{m\in\mathbb{Z}_*} e^{-2m^2|\check{\mathbf{k}}|^2 t} im|\check{\mathbf{k}}| \Big[J_{\mathbf{k}}\widehat{\mathbf{u}}_{0,m\mathbf{k}} \cdot \overline{\widehat{\mathbf{u}}}_{0,m\mathbf{k}} \Big] \\ &\quad - L_1L_2L_3 \sum_{m\in\mathbb{Z}_*} e^{-2m^2|\check{\mathbf{k}}|^2 t} im|\check{\mathbf{k}}| \Big[2\cos(\widetilde{k}_3\Omega t)\sin(\widetilde{k}_3\Omega t)|\widehat{\mathbf{u}}_{0,m\mathbf{k}}|^2 \Big].\end{aligned}$$

When summing over m and $-m$, the last sum vanishes. Hence,

$$\mathcal{H}(t) = L_1L_2L_3 \sum_{m\in\mathbb{Z}_*} e^{-2m^2|\check{\mathbf{k}}|^2 t} 2m|\check{\mathbf{k}}| \Big[\big(\mathrm{Re}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \times \mathrm{Im}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \big) \cdot \widetilde{\mathbf{k}} \Big].$$

Thus, $\mathcal{H}(t) = 0$ for all $t \geq 0$, provided that

$$\big(\mathrm{Re}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \times \mathrm{Im}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \big) \cdot \widetilde{\mathbf{k}} = 0, \text{ for all } m \in \mathbb{Z}_*. \tag{5.11}$$

However, since $\widehat{\mathbf{u}}_{0,m\mathbf{k}}$ is orthogonal to $\widetilde{\mathbf{k}}$ then $\mathrm{Re}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \times \mathrm{Im}(\widehat{\mathbf{u}}_{0,m\mathbf{k}})$ is co-linear with $\widetilde{\mathbf{k}}$, and as result (5.11) is equivalent to

$$\mathrm{Re}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) \times \mathrm{Im}(\widehat{\mathbf{u}}_{0,m\mathbf{k}}) = 0, \text{ for all } m \in \mathbb{Z}_*. \tag{5.12}$$

This class of solutions (5.10), (5.12) with vanishing helicity for the NSE of rotating fluids has more restrictive wave vectors, i.e., the $m\mathbf{k}$'s in (5.10), than those studied in [7, Proposition 6.4] for the NSE without the rotation.

Corollary 5.3. *There exist infinitely many vector spaces of infinite dimensions that are invariant under the NSE* (1.16)*, and each space is not a subspace of any of the others.*

Proof. According to Theorem 5.1, each vector space $V_{\mathbf{k}}$ is invariant under the NSE (1.16), and each has infinite dimension. Moreover, there are infinite many $\mathbf{k}$'s which are pairwise not co-linear. For those $\mathbf{k}$'s, the corresponding $V_{\mathbf{k}}$'s are the vector spaces for which the statement holds true. □

Next, we consider the case of non-zero spatial average solutions.

Theorem 5.4. *Suppose $u_0 = \mathbf{U}_0 + w_0$, where $\mathbf{U}_0$ is a constant vector in $\mathbb{R}^3$, and*

$$w_0 = \sum_{m\in\mathbb{Z}_*} \widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}} \in V_{\mathbf{k}}.$$

Let $\mathbf{U}(t) = e^{-\Omega tJ}\mathbf{U}_0$ *and* $\mathbf{V}(t) = \int_0^t \mathbf{U}(\tau)d\tau$. *Define, for* $\mathbf{x} \in \mathbb{R}^3$ *and* $t > 0$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(t) + \sum_{m\in\mathbb{Z}_*} e^{-m^2|\check{\mathbf{k}}|^2 t} e^{-im\check{\mathbf{k}}\cdot\mathbf{V}(t)} E_{\mathbf{k}}(-\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}, \tag{5.13}$$

$$\begin{aligned} p(\mathbf{x}, t) = p_*(t) - \Omega \sum_{m\in\mathbb{Z}_*} &\frac{i}{m|\check{\mathbf{k}}|} e^{-m^2|\check{\mathbf{k}}|^2 t} e^{-im\check{\mathbf{k}}\cdot\mathbf{V}(t)} \\ &\cdot \left[\cos(\widetilde{k}_3\Omega t)J\widetilde{\mathbf{k}} + \sin(\widetilde{k}_3\Omega t)\mathbf{e}_3\right] \cdot \widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}, \end{aligned} \tag{5.14}$$

where $p_*(t)$ *is any scalar function. Then* $(\mathbf{u}, p)$ *is a solution of* (1.1) *and* (1.2) *on* $\mathbb{R}^3 \times (0, \infty)$.

Proof. Let $w(t) = e^{-tA}e^{-t\Omega S}w_0$, which we write as $w(t) = \mathbf{w}(\cdot, t)$. By Theorem 5.1, particularly, (5.7) and (5.9), we have

$$(\mathbf{w}\cdot\nabla)\mathbf{w} = 0 \text{ and } \mathbf{w}_t - \Delta\mathbf{w} + \Omega J\mathbf{w} = -\nabla q,$$

where the scalar function $q(\mathbf{x}, t)$ satisfies the geostrophic balance

$$\Omega \mathrm{div}(J\mathbf{w}) = -\Delta q.$$

We solve this equation by

$$q(\mathbf{x}, t) = p_*(t) + \Omega(-\Delta)^{-1}\mathrm{div}(J\mathbf{w}(\mathbf{x}, t)), \tag{5.15}$$

where the inverse operator $(-\Delta)^{-1}$ is meant to apply to functions having zero spatial average over $\mathbb{T}_{\mathbf{L}}$.

We note that $\mathbf{w}(\mathbf{x}, t)$ is Gevrey-regular, for each $t > 0$. Therefore, the following calculations are valid in the classical sense. Define

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{w}(\mathbf{x} - \mathbf{V}(t), t) + \mathbf{U}(t) \text{ and } p(\mathbf{x}, t) = q(\mathbf{x} - \mathbf{V}(t), t). \tag{5.16}$$

(The functions $\mathbf{u}$ and p in (5.16) will be proved to agree with those in (5.13) and (5.14) later.)

Since $\mathbf{w}$ is divergence-free, then, clearly, so is $\mathbf{u}$. We calculate, with the shorthand notation $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\mathbf{w} = \mathbf{w}(\mathbf{x} - \mathbf{V}(t), t)$,

$$\begin{aligned} \mathbf{u}_t &= \mathbf{w}_t - (\mathbf{U}\cdot\nabla)\mathbf{w} + \mathbf{U}' = \mathbf{w}_t - (\mathbf{U}\cdot\nabla)\mathbf{w} - \Omega J\mathbf{U}, \\ (\mathbf{u}\cdot\nabla)\mathbf{u} &= [(\mathbf{w} + \mathbf{U})\cdot\nabla]\mathbf{w} = [\mathbf{U}\cdot\nabla]\mathbf{w}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Big[\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \Omega J\mathbf{u}\Big]_{(\mathbf{x},t)} &= \Big[\mathbf{w}_t - \Delta\mathbf{w} + \Omega J\mathbf{w}\Big]_{(\mathbf{x}-\mathbf{V}(t),t)} \\ &= -\Big[\nabla q\Big]_{(\mathbf{x}-\mathbf{V}(t),t)} = -\Big[\nabla p\Big]_{(\mathbf{x},t)}. \end{aligned}$$

We conclude that $(\mathbf{u}, p)$ defined by (5.16) is a solution of (1.1) and (1.2) on $\mathbb{R}^3 \times (0, \infty)$. It remains to calculate them explicitly. First, we have

$$\mathbf{w}(\mathbf{x}, t) = \sum_{m\in\mathbb{Z}_*} e^{-m^2|\check{\mathbf{k}}|^2 t} E_{\mathbf{k}}(-\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{5.17}$$

Then, thanks to (5.16) and (5.17), we obtain formula (5.13) for $\mathbf{u}(\mathbf{x}, t)$. Next, we calculate $\mathrm{div}(J\mathbf{w})$ by

$$\mathrm{div}(J\mathbf{w}(\mathbf{x}, t)) = \sum_{m\in\mathbb{Z}_*} e^{-m^2|\check{\mathbf{k}}|^2 t} i\, m\check{\mathbf{k}}^{\mathrm{T}} E_{\mathbf{k}}(-\widetilde{k}_3\Omega t)\widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{5.18}$$

Denote $\mathbf{z} = \widehat{\mathbf{u}}_{0,m\mathbf{k}} \in X_{\mathbf{k}}$, then

$$\begin{aligned}\check{\mathbf{k}}^{\mathrm{T}} J E_{\mathbf{k}}(t)\mathbf{z} &= \cos(t)|\check{\mathbf{k}}|\widetilde{\mathbf{k}} \cdot J\mathbf{z} + \sin(t)|\check{\mathbf{k}}|\widetilde{\mathbf{k}} \cdot (\mathbf{e}_3 \times J_{\mathbf{k}}\mathbf{z}) \\ &= |\check{\mathbf{k}}|\big[\cos(t)(J^{\mathrm{T}}\widetilde{\mathbf{k}}) \cdot \mathbf{z} - \sin(t)\mathbf{e}_3 \cdot J_{\mathbf{k}}^2\mathbf{z}\big] \\ &= |\check{\mathbf{k}}|[-\cos(t)J\widetilde{\mathbf{k}} + \sin(t)\mathbf{e}_3] \cdot \mathbf{z}.\end{aligned}$$

(We used the fact J is anti-symmetric and relation (A.2) below.) Thus, together with (5.15) and (5.18),

$$q(\mathbf{x},t) = p_*(t) - \Omega \sum_{m\in\mathbb{Z}_*} \frac{i\, e^{-m^2|\check{\mathbf{k}}|^2 t}}{m|\check{\mathbf{k}}|}\big[\cos(\widetilde{k}_3\Omega t)J\widetilde{\mathbf{k}} + \sin(\widetilde{k}_3\Omega t)\mathbf{e}_3\big] \cdot \widehat{\mathbf{u}}_{0,m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}}. \tag{5.19}$$

From (5.16) and (5.19), we obtain formula (5.14) for $p(\mathbf{x}, t)$. The proof is complete. □

Remark 5.5. Let $\mathbf{k}$ and u_0 be as in Theorem 5.1. We rewrite (5.6) as

$$u(t) = \sum_{n=1}^{\infty} Q_{n,\Omega}(t)e^{-\Lambda_n t}, \tag{5.20}$$

where $Q_{n,\Omega}(t) = e^{-\Omega t S} R_{\Lambda_n} u_0$. By (2.14), $Q_{n,\Omega} \in \mathcal{F}_1(\mathcal{V})$. Therefore, (5.20) is the asymptotic expansion of $u(t)$.

By (1.22) and (5.2), we have either $Q_{n,\Omega} = 0$, or there is a unique number $m \in \mathbb{N}$ with $m^2|\check{\mathbf{k}}|^2 = \Lambda_n$ and

$$Q_{n,\Omega}(t) = (\cos(\widetilde{k}_3\Omega t)I_3 - \sin(\widetilde{k}_3\Omega t)J_{\mathbf{k}})[\widehat{\mathbf{u}}_{m\mathbf{k}} e^{im\check{\mathbf{k}}\cdot\mathbf{x}} + \widehat{\mathbf{u}}_{-m\mathbf{k}} e^{-im\check{\mathbf{k}}\cdot\mathbf{x}}]. \tag{5.21}$$

In case $k_3 = 0$, we then have $Q_{n,\Omega}(t) = R_{\Lambda_n} u_0$ which is independent of Ω.

Consider the case $k_3 \neq 0$. Given $T > 0$, define the time averaging function

$$\bar{Q}_{n,\Omega}(t) = \frac{1}{T}\int_t^{t+T} Q_{n,\Omega}(\tau)d\tau. \tag{5.22}$$

One can see from (5.21) and (5.22) that

$$\lim_{\Omega\to\pm\infty} \bar{Q}_{n,\Omega}(t) = 0, \quad \text{for any } t \in \mathbb{R}.$$

Appendix A.

Proof of (1.22). First, we have from (1.17) that

$$e^{tS}u = \sum\nolimits' e^{tS_{\mathbf{k}}}\widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \tag{A.1}$$

where $S_{\mathbf{k}} = \widehat{P}_{\mathbf{k}} J \widehat{P}_{\mathbf{k}}$. Using formula (1.9) for $\widehat{P}_{\mathbf{k}}$ and the fact $\widetilde{\mathbf{k}}^{\mathrm{T}} J\widetilde{\mathbf{k}} = 0$, we can compute

$$S_{\mathbf{k}} = J - \widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} J - J\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} + \widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} J\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} = J - \widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} J + (\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^{\mathrm{T}} J)^{\mathrm{T}}.$$

With $|\widetilde{\mathbf{k}}|^2 = 1$, we have

$$S_{\mathbf{k}} = \begin{pmatrix} 0 & -1 + \widetilde{k}_1^2 + \widetilde{k}_2^2 & \widetilde{k}_3\widetilde{k}_2 \\ 1 - \widetilde{k}_1^2 - \widetilde{k}_2^2 & 0 & -\widetilde{k}_3\widetilde{k}_1 \\ -\widetilde{k}_3\widetilde{k}_2 & \widetilde{k}_3\widetilde{k}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{k}_3^2 & \widetilde{k}_3\widetilde{k}_2 \\ \widetilde{k}_3^2 & 0 & -\widetilde{k}_3\widetilde{k}_1 \\ -\widetilde{k}_3\widetilde{k}_2 & \widetilde{k}_3\widetilde{k}_1 & 0 \end{pmatrix} = \widetilde{k}_3 J_{\mathbf{k}}.$$

Let $\mathbf{z} \in X_{\mathbf{k}}$. Then

$$J_{\mathbf{k}}^2\mathbf{z} = \widetilde{\mathbf{k}} \times (\widetilde{\mathbf{k}} \times \mathbf{z}) = (\widetilde{\mathbf{k}} \cdot \mathbf{z})\widetilde{\mathbf{k}} - (\widetilde{\mathbf{k}} \cdot \widetilde{\mathbf{k}})\mathbf{z} = -\mathbf{z}. \tag{A.2}$$

We observe

$$\frac{d}{dt}\big(E_{\mathbf{k}}(t)\mathbf{z}\big) = -(\sin t)\mathbf{z} + (\cos t)J_{\mathbf{k}}\mathbf{z} = (\sin t)J_{\mathbf{k}}^2\mathbf{z} + (\cos t)J_{\mathbf{k}}\mathbf{z},$$

thus,

$$\frac{d}{dt}\big(E_{\mathbf{k}}(t)\mathbf{z}\big) = J_{\mathbf{k}}\big(E_{\mathbf{k}}(t)\mathbf{z}\big).$$

This linear ODE yields the solution $E_{\mathbf{k}}(t)\mathbf{z} = e^{tJ_{\mathbf{k}}}E_{\mathbf{k}}(0)\mathbf{z} = e^{tJ_{\mathbf{k}}}\mathbf{z}$. Therefore,

$$e^{tS_{\mathbf{k}}}\mathbf{z} = e^{\widetilde{k}_3 tJ_{\mathbf{k}}}\mathbf{z} = E_{\mathbf{k}}(\widetilde{k}_3 t)\mathbf{z}. \tag{A.3}$$

Letting $\mathbf{z} = \widehat{\mathbf{u}}_{\mathbf{k}}$, we obtain (1.22) thanks to (A.1) and (A.3). ☐

Proof of Lemma 2.8. Denote

$$I(t) = \begin{pmatrix} e^{\alpha t}\cos(\omega t) \\ e^{\alpha t}\sin(\omega t)\end{pmatrix} \text{ and } D_{-1} = \frac{1}{\alpha^2+\omega^2}\begin{pmatrix} \alpha & \omega \\ -\omega & \alpha\end{pmatrix}.$$

First, we prove, for all $m \in \mathbb{N}\cup\{0\}$, that

$$\int t^m I(t)dt = \sum_{n=0}^{m} \frac{(-1)^{m-n}m!}{n!} t^n (D_{-1})^{m+1-n} I(t) + \mathbf{C}, \tag{A.4}$$

where $\mathbf{C}$ denotes an arbitrary constant vector in $\mathbb{R}^2$. Indeed, it is well-known that

$$\int I(t)dt = D_{-1}I(t) + \mathbf{C}, \tag{A.5}$$

which proves (A.4) for $m = 0$. For $m \in \mathbb{N}$, integration by parts, with the use of (A.5), yields

$$\int t^m I(t)dt = t^m D_{-1}I(t) - mD_{-1}\int t^{m-1}I(t)dt + \mathbf{C}.$$

By iterating this recursive relation, we obtain (A.4). Then the statement of Lemma 2.8 obviously follows from (A.4). ☐

Lemma A.1. *Let $m \in \mathbb{N}$ and $a_n, b_n \in \mathbb{C}$, $\omega_n \in \mathbb{R}$ for $1 \le n \le m$. Suppose the function $f : \mathbb{R} \to \mathbb{C}$ defined by*

$$f(t) = \sum_{n=1}^{m}[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)], \text{ for } t \in \mathbb{R}, \tag{A.6}$$

satisfies

$$\lim_{t\to\infty} f(t) = 0. \tag{A.7}$$

Then $f(t) = 0$ for all $t \in \mathbb{R}$.

Proof. By considering the real and imaginary parts of f, we can assume, without loss of generality, that $a_n, b_n \in \mathbb{R}$ for all $n = 1, \ldots, m$. Equation (A.6) can be rewritten so that ω_n's are strictly increasing non-negative numbers. We then convert (A.6), with some re-indexing, to the following form

$$f(t) = A_0 + \sum_{n=1}^{N} A_n\cos(\omega_n t + \varphi_n), \tag{A.8}$$

where $N \ge 0$, A_0, A_n and φ_n, for $1 \le n \le N$, are constants in $\mathbb{R}$, and ω_n's are positive, strictly increasing in n.

Claim: $A_n = 0$ for all $0 \le n \le N$.

With this Claim, we have $f = 0$ as desired. We now prove the Claim by induction in N.

Case $N = 0$. Then $f(t) = A_0$, which, by (A.7), yields $A_0 = 0$. Therefore, the Claim is true for $N = 0$.

Let $N \ge 0$. Assume the Claim is true for any function of the form (A.8) that satisfies (A.7). Now, suppose function

$$f(t) = A_0 + \sum_{n=1}^{N+1} A_n \cos(\omega_n t + \varphi_n) \tag{A.9}$$

satisfies (A.7), with positive numbers ω_n's being strictly increasing in n.

Set $T = 2\pi/\omega_{N+1} > 0$, and define function $g(t) = \int_t^{t+T} f(\tau)d\tau$.

On the one hand, we have, for $1 \le n \le N$, that

$$\begin{aligned}\int_t^{t+T} \cos(\omega_n \tau + \varphi_n)d\tau &= \frac{2}{\omega_n} \sin(\omega_n T/2)\cos(\omega_n t + \varphi_n + \omega_n T/2)\\ &= D_n \cos(\omega_n t + \varphi_n'),\end{aligned}$$

where $D_n = 2\omega_n^{-1}\sin(\omega_n \pi/\omega_{N+1}) > 0$ and number $\varphi_n' \in \mathbb{R}$. On the other hand,

$$\int_t^{t+T} \cos(\omega_{N+1}\tau + \varphi_{N+1})d\tau = 0.$$

Hence,

$$g(t) = A_0 T + \sum_{n=1}^{N} A_n D_n \cos(\omega_n t + \varphi_n').$$

Moreover, it follows from (A.7) that $g(t) \to 0$, as $t \to \infty$. By the induction hypothesis applied to function g, we obtain $A_0 T = 0$ and $A_n D_n = 0$, for $1 \le n \le N$. Thus, $A_n = 0$, for $0 \le n \le N$, and (A.9) becomes

$$f(t) = A_{N+1}\cos(\omega_{N+1} t + \varphi_{N+1}).$$

This form of f and property (A.7) imply $A_{N+1} = 0$. Therefore, the Claim holds true for $N + 1$. By the induction principle, it is true for all $N \ge 0$. □

Acknowledgments

The work of E.S.T. was supported in part by the Einstein Stiftung/Foundation - Berlin, through the Einstein Visiting Fellow Program, and by the John Simon Guggenheim Memorial Foundation. The authors would like to thank Ciprian Foias for his insights, inspiring and stimulating discussions.

References

[1] Babin, A., Mahalov, A., and Nicolaenko, B. Global regularity of 3D rotating Navier-Stokes equations for resonant domains. *Indiana Univ. Math. J. 48*, 3 (1999), 1133–1176.

[2] Cao, D., and Hoang, L. Asymptotic expansions in a genral system of decaying functions for the Navier-Stokes equations. *Annali di Matematica Pura ed Applicata* (2019), 50 pp. in press. doi:10.1007/s10231-019-00911-3.

[3] Cao, D., and Hoang, L. Long-time asymptotic expansions for Navier-Stokes equations with power-decaying forces. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* (2019), 38 pp. in press. doi:10.1017/prm.2018.154.

[4] Chemin, J.-Y., Desjardins, B., Gallagher, I., and Grenier, E. *Mathematical geophysics*, vol. 32 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, Oxford, 2006. An introduction to rotating fluids and the Navier-Stokes equations.

[5] Constantin, P., and Foias, C. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[6] Foias, C. *Navier-Stokes equations*. Texas A&M lecture notes, 2002. (unpublished).

[7] Foias, C., Hoang, L., and Nicolaenko, B. On the helicity in 3D-periodic Navier-Stokes equations. I. The non-statistical case. *Proc. Lond. Math. Soc. (3) 94*, 1 (2007), 53–90.

[8] Foias, C., Hoang, L., and Nicolaenko, B. On the helicity in 3D-periodic Navier-Stokes equations. II. The statistical case. *Comm. Math. Phys. 290*, 2 (2009), 679–717.

[9] Foias, C., Hoang, L., Olson, E., and Ziane, M. On the solutions to the normal form of the Navier-Stokes equations. *Indiana Univ. Math. J. 55*, 2 (2006), 631–686.

[10] Foias, C., Hoang, L., Olson, E., and Ziane, M. The normal form of the Navier-Stokes equations in suitable normed spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire 26*, 5 (2009), 1635–1673.

[11] Foias, C., Hoang, L., and Saut, J.-C. Asymptotic integration of Navier-Stokes equations with potential forces. II. An explicit Poincaré-Dulac normal form. *J. Funct. Anal. 260*, 10 (2011), 3007–3035.

[12] Foias, C., Hoang, L., and Saut, J.-C. Navier and Stokes meet Poincaré and Dulac. *J. Appl. Anal. Comput. 8*, 3 (2018), 727–763.

[13] Foias, C., Manley, O., Rosa, R., and Temam, R. *Navier-Stokes equations and turbulence*, vol. 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[14] Foias, C., and Saut, J.-C. Asymptotic behavior, as $t \to \infty$, of solutions of the Navier-Stokes equations. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982)*, vol. 84 of *Res. Notes in Math.* Pitman, Boston, MA, 1983, pp. 74–86.

[15] Foias, C., and Saut, J.-C. Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. *Indiana Univ. Math. J. 33*, 3 (1984), 459–477.

[16] Foias, C., and Saut, J.-C. On the smoothness of the nonlinear spectral manifolds associated to the Navier-Stokes equations. *Indiana Univ. Math. J. 33*, 6 (1984), 911–926.

[17] Foias, C., and Saut, J.-C. Linearization and normal form of the Navier-Stokes equations with potential forces. *Ann. Inst. H. Poincaré Anal. Non Linéaire 4*, 1 (1987), 1–47.

[18] Foias, C., and Saut, J.-C. Asymptotic integration of Navier-Stokes equations with potential forces. I. *Indiana Univ. Math. J. 40*, 1 (1991), 305–320.

[19] Foias, C., and Temam, R. Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal. 87*, 2 (1989), 359–369.

[20] Hoang, L. T., and Martinez, V. R. Asymptotic expansion in Gevrey spaces for solutions of Navier-Stokes equations. *Asymptot. Anal. 104*, 3–4 (2017).

[21] Hoang, L. T., and Martinez, V. R. Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces. *J. Math. Anal. Appl. 462*, 1 (2018), 84–113.

[22] Ladyzhenskaya, O. A. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach Science Publishers, New York, 1969.

[23] Moffatt, H. K. The degree of knottedness of tangled vortex lines. *J. Fluid Mech. 36* (1969), 117–129.

[24] Moffatt, H. K., and Tsinober, A. Helicity in laminar and turbulent flow. In *Annual review of fluid mechanics, Vol. 24*. Annual Reviews, Palo Alto, CA, 1992, pp. 281–312.

[25] Shi, Y. A Foias-Saut type of expansion for dissipative wave equations. *Comm. Partial Differential Equations 25*, 11-12 (2000), 2287–2331.

[26] Temam, R. *Navier-Stokes equations and nonlinear functional analysis*, second ed., vol. 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

[27] Temam, R. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[28] Vanyo, J. *Rotating Fluids in Engineering and Science*. Dover Civil and Mechanical Engineering. Dover Publications, 2001. Reprint of the Butterworth-Heinemann, Boston, 1993 edition.

[1] Department of Mathematics and Statistics, Texas Tech University, 1108 Memorial Circle, Lubbock, TX 79409-1042, U.S.A.
E-mail address: luan.hoang@ttu.edu

[2] Department of Mathematics, 3368 TAMU, Texas A&M University, College Station, TX 77843-3368, U.S.A.

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, U.K.

Weizmann Institute of Science, Department of Computer Science and Applied Mathematics, P.O. Box 26, Rehovot, 76100, Israel
E-mail address: titi@math.tamu.edu and Edriss.Titi@damtp.cam.ac.uk